American Journal of Engineering Research (AJER)
e-ISSN: 2320-0847 p-ISSN : 2320-0936
Volume-6, Issue-7, pp-216-219
www.ajer.org
Research Paper Open Access

Differential Equations of Non-integer Order with Integral Boundary Conditions

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ABSTRACT: *In this paper, we apply the comparison result without locally Holder continuity due to Vasundhara Devi to develop monotone method for the problem and obtain existence and uniqueness of solution of the differential equation of non-integer order with integral boundary conditions.*

Keywords: *Fractional differential equation, integral boundary conditions, lower and upper solutions, existence and uniqueness.*
Subject Classification: *34A12, 34A99, 34C60*

Date of Submission: 27 -06-2017 Date of acceptance: 20-07-2017

I. INTRODUCTION

The theory of fractional differential equations has become an important area of investigation because of its wide applications in many branches of sciences, engineering, nature and social sciences. Lakshmikantham and Vatsala [12, 14] obtained local and global existence of solutions of Riemann-Liouville fractional differential equations and uniqueness of solutions. Monotone method for Riemann-Liouville fractional differential equations with initial conditions is developed by McRae [17] involving study of qualitative properties of solutions of initial value problem. Jankwoski [7] formulated some comparison results and obtained existence and uniqueness of solution of differential equations with integral boundary conditions.

In 2009, Wang and Xie [22] developed monotone method and obtained existence and uniqueness of solution of fractional differential equation with integral boundary condition. Basic theory of fractional differential equations in Banach spaces is well established by Lakshmikantham in [10, 11]. Vasundhara Devi developed [3] the general monotone method for periodic boundary value problem of Caputo fractional differential equation when the function is sum of non-decreasing and non-increasing function. The Caputo fractional differential equation with periodic boundary conditions has been studied by present authors [5, 6] and developed monotone method for the problem. Existence and uniqueness of solution of Riemann-Liouville fractional differential equation with integral boundary conditions is also obtained by Nanware and Dhaigude in [18, 19, 20]. The qualitative properties of solutions such as existence, periodicity, ergodicity, almost periodic and pseudo-almost periodic etc. of fractional differential equations and fractional integro-differential equations were studied by many researchers. For more details see [1, 2, 4, 8, 9, 13, 15, 16 , 21].

In this paper, we consider system of differential equations of non-integer order with integral boundary conditions and develop monotone method for system of differential equations of non-integer order with integral boundary conditions and obtained existence and uniqueness of solution of the problem.

The paper is organized in the following manner: In section 2, we consider some definitions and lemmas required in next section. In section 3, monotone method is developed for the problem. As an application of the method existence and uniqueness results for system of differential equations of non-integer order with integral boundary conditions are obtained.

II. PRILIMINARIES

In 2009, Wang and Xie [22] developed monotone iterative method for the following fractional differential equations with integral boundary conditions with Holder continuity and obtained existence and uniqueness of solution of the problem

$$D^q u(t) = f(t,u), \qquad t \in J = [0,T], \quad T \geq 0$$

$$u(t) = \lambda \int_0^T u(s)ds + d, \quad d \in R. \tag{2.0}$$

where $0 < q < 1$, λ is 1 or -1 and $f \in C[J \times R, R]$, D^q is the Riemann-Liouville fractional derivative of non-integer order derivative of order q.

In this paper, we consider the following system of differential equations of non-integer order with integral boundary conditions

$$D^q u_i(t) = f_i(t, u_1(t), u_2(t)), \qquad t \in J = [0, T], \quad T \geq 0$$
$$u_i(t) = \int_0^T u_i(s)ds + d_i, \quad d_i \in R, \quad i = 1,2. \tag{2.1}$$

where f_1, f_2 in $C[J \times R^2, R]$, $\lambda = 1$, $0 < q < 1$.

We develop monotone method for the problem (2.1) for the class of continuous functions and study existence and uniqueness of solutions of the problem (2.1).

Lemma 2.1 [4] Let $m \in C_p([t_0, T], R)$ and for any $t_1 \in (t_0, T]$ we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 < t < t_1$. Then $D^q m(t_1) \geq 0$.

Lemma 2.2 [12] Let $\{u_\epsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for each $\epsilon > 0$ where $D^q u_\epsilon(t) = f(t, u_\epsilon(t))$, $u_\epsilon(t_0) = u_\epsilon(t)(t - t_0)^{1-q}]_{t=t_0}$ and
$|f(t, u_\epsilon(t))| \leq M$ for $t_0 \leq t < T$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous.

Theorem 2.1 [22] Assume that
i) *v(t)* and *w(t)* in $C_p(J, R)$ are ower and upper solutions of (2.1)
ii) $f(t, u(t))$ satisfy one-sided Lipschitz condition
$f(t, u) - f(t, v) \leq L(u - v)$, $\quad L \in \left(0, \frac{1}{\Gamma(1-q)T^q}\right)$
Then *v(0)≤ w(0)* implies that *v(t)≤ w(t)*, $0 \leq t \leq T$

Definition 2.1. A pair of functions $v(t) = (v_1, v_2)$ and $w(t) = (w_1, w_2)$ in $C_p(J, R)$ are said to be lower and upper solutions of the problem (2.1) if

$$D^q v_i(t) \leq f_i(t, v_1(t), v_2(t)), \qquad v_i(0) \leq \int_0^T v_i(s)ds + d_i$$
$$D^q w_i(t) \geq f_i(t, w_1(t), w_2(t)), \qquad w_i(0) \geq \int_0^T w_i(s)ds + d_i.$$

III. MONOTONE METHOD

In this section we develop monotone method for the problem (2.1) and obtain the existence and uniqueness of solutions of the problem (2.1).

Definition 3.1 A function $f_i = (t, u_1(t), u_2(t))$ in $C_p(J \times R^2, R)$ is said to be quasi-monotone non-decreasing if

$$f_i(t, u_1(t), u_2(t)) \leq f_i(t, v_1(t), v_2(t)), \quad if \quad u_i = v_i \; and \; u_j \leq v_j, \; i \neq j, \quad i = j = 1,2.$$

Definition 3.2 A pair of functions $v(t) = (v_1, v_2)$ and $w(t) = (w_1, w_2)$ in $C_p(J, R)$ are said to be weakly coupled lower and upper solutions of the problem (2.1) if

$$D^q v_i(t) \leq f_i(t, v_1(t), v_2(t)), \qquad v_i(0) \leq \int_0^T w_i(s)ds + d_i$$
$$D^q w_i(t) \geq f_i(t, w_1(t), w_2(t)), \qquad w_i(0) \geq \int_0^T v_i(s)ds + d_i.$$

Theorem 3.1 Assume that
i) $f_i(t, v_1(t), v_2(t))$ is quasi-monotone non-decreasing,
ii) $v_o(t)$ and $w_o(t)$ in $C_p(J, R)$ are weakly coupled lower and upper solutions of (2.1) such that $v_0(t) \leq w_0(t)$ on *J=[0,T]*
iii) $f(t, u(t))$ satisfy one-sided Lipschitz condition

$f(t,u) - f(t,v) \le -L(u-v), \qquad L \ge 0$

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J, \mathrm{R})$ such that $\{v_n(t)\} \square\ v(t)$ and $\{w_n(t)\} \square\ w(t)$ as $n \to \infty$ where $v(t)$ and $w(t)$ are minimal and maximal solutions of (2.1) respectively.

Proof. For any $\eta(t) = (\eta_1(t), \eta_2(t))$ and $\mu(t) = (\mu_1(t), \mu_2(t))$ in $C_p(J, \mathrm{R})$ such that for $v_i^0(0) \le \eta_i$ and $w_i^0(0) \le \mu_i$ on $\boldsymbol{J}$, consider the following linear fractional differential equation

$$D^q u_i(t) + M_i u_i(t) = f_i(t, \eta_1(t), \eta_2(t)) - M_i \eta_i(t), \qquad u_i(0) = \int_0^T u_i(s)ds + d_i, \ \ i = 1,2\,. \qquad (3.1)$$

Uniqueness of solution of linear fractional differential equation (3.1) can be proved as in [18].
Define a mapping A by $[\eta_i(t), \mu_i(t)] = u_i(t)$, where $u_i(t)$ is the unique solution of the problem (3.1). This mapping generates the sequences $\{v_i^n(t)\}$ *and* $\{w_i^n(t)\}$. Now we prove that

(I) $v^0 \le A[v^0, w^0], \quad w^0 \ge A[w^0, v^0]$

(II) A possesses the monotone property on the segment

$[v^0, w^0] = \{(u_1, u_2) \in C[J, R]: v_i^0 \le u_i \le w_i^0\}, \ \ i = 1,2.$

Set $A\,[v^0, w^0] = v^1(t)$, where $v^1(t) = (v_1^1 v_2^1)$ is the unique solution of te problem (3.1) with $\eta_i = v_i^0(0)$.
Setting $p_i(t) = v_i^0(t) - v_i^1(t)$ we see that

$$\begin{aligned} D^q p_i(t) &\le f_i(t, v_1^0(t), v_2^0(t)) - f_i(t, v_1^1(t), v_2^1(t)) \\ &\le -M_i p_i(t) \end{aligned}$$

and $p_i(0) \le 0$.

Applying Theorem 2.1, we get $p_i(t) \le 0$ on $0 \le t \le T$ and hence $v_i^0(t) - v_i^1(t) \le 0$ which implies $v_i^0 \le A[v^0, w^0]$.
Set $A[v^0, w^0] = w^1(t)$, where $w^1(t) = (w_1^1 w_2^1)$ is the unique solution of the problem (3.1) with $\mu_i = w_i^0(0)$. Setting $p_i(t) = w_i^0(t) - w_i^1(t)$ we see that

$$\begin{aligned} D^q p_i(t) &\ge f_i(t, w_1^0(t), w_2^0(t)) - f_i(t, w_1^1(t), w_2^1(t)) \\ &\ge -M_i p_i(t) \end{aligned}$$

and $p_i(0) \ge 0$. Applying Theorem 2.1, we have $w_i^0 \ge w_i^1$. Hence $w^0 \ge A[w^0, v^0]$. This proves (I).
Let $\eta(t), \beta(t), \mu(t) \in [v^0, w^0]$ with $\eta(t) \le \beta(t)$. Suppose that $A[\eta, \mu] = u(t), A[\beta, \mu] = v(t)$. Then setting $p_i(t) = u_i(t) - v_i(t)$ we find that

$$D^q p_i(t) \le -M_i p_i(t)$$

and $p_i(0) \le 0.$

As before in (I), we have $A[\eta, \mu] \le A[\beta, \mu]$. Similarly we can prove that $A[\eta, v] \le A[\eta, \mu]$.Thus the mapping A possesses monotone property on the segment $[v^0, w^0]$. Now in view of (I) and (II), define the sequences $v_i^n(t) = A[v_i^{n-1}, w_i^{n-1}]$, $\quad w_i^n(t) = A[w_i^{n-1}, v_i^{n-1}]$ on the segment $[v^0, w^0]$ by

$$D^q v_i^n(t) = f_i\left(t, v_i^{n-1}(t), v_2^{n-1}(t)\right) - M_i[v_i^n - v_i^{n-1}], \quad v_i^0(0) = \int_0^T v_i^{n-1}(s)ds + d_i$$

$$D^q w_i^n(t) = f_i\left(t, w_i^{n-1}(t), w_2^{n-1}(t)\right) - M_i[w_i^n - w_i^{n-1}], \quad w_i^0(0) = \int_0^T w_i^{n-1}(s)ds + d_i$$

From (I) , we have $v_i^0 \le v_i^1, \quad w_i^0 \ge w_i^1$. Assume that $v_i^{k-1} \le v_i^k, \quad w_i^{k-1} \ge w_i^k$.
To prove $v_i^k \le v_i^{k+1}, \quad w_i^k \ge w_i^{k+1}$ *and* $\quad v_i^k \ge w_i^k$, define $p_i(t) = v_i^k(t) - v_i^{k+1}(t)$.
Thus

$$D^q p_i(t) \le -M_i p_i(t)$$

and $p_i(0) \le 0.$

It follows from Theorem 2.1 that $p_i(t) \le 0$, which gives $v_i^k(t) \le v_i^{k+1}(t)$. Similarly, we prove $w_i^k(t) \ge w_i^{k+1}(t)$ *and* $v_i^k(t) \ge w_i^k(t)$. By induction it follows that

$$v_i^0(t) \le v_i^1(t) \le v_i^2(t) \le \dots \le v_i^n(t) \le w_i^n(t) \le v_i^{n-1}(t) \le \cdots \le w_i^1(t) \le w_i^0(t).$$

Thus the sequences $\{v^n(t)\}$ and $\{w^n(t)\}$ are bounded from below and bounded from above respectively and monotonically non-decreasing and monotonically non-increasing on J. Hence point-wise limit exist and are given by $\lim_{n\to\infty} v_i^n(t) = v_i(t)$, $\lim_{n\to\infty} w_i^n(t) = w_i(t)$ on J.
Using corresponding fractional Volterra integral equations

$$v_i^n(t) = v_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \{f(s, v_1^n(s), v_2^n(s)\,) - M(v_i^n - v_i^{n-1})\} ds$$

$$w_i^n(t) = w_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \{f\left(s, w_1^n(s), w_2^n(s)\right) - M(w_i^n - w_i^{n-1})\} ds$$

it follows that $v(t)$ and $w(t)$ are solutions of (3.1).
Next we claim that $v(t)$ and $w(t)$ are the minimal and maximal solutions of (2.1). Let $u(t)$ be any solution of (2.1) different from $v(t)$ and $w(t)$**,** so that there exists $\boldsymbol{k}$ such that $v_i^k(t) \le u_i(t) \le w_i^k(t)$ on $\boldsymbol{J}$ and set $p(t) = v_i^{k+1}(t) - u_i(t)$ so that

$$D^q p_i(t) \ge -M_i p_i(t)$$

and $\quad p(0) \geq 0$.

Thus $v_i^{k+1}(t) \leq u_i(t)$ on *J*. Since $v_i^0(t) \leq u_i(t)$ on *J*, by induction it follows that $v_i^k(t) \leq u_i(t)$ for all ***k***. Similarly we can prove $u_i(t) \leq w_i^k(t)$ on ***J***. Thus $v_i^k(t) \leq u_i(t) \leq w_i^k(t)$ on ***J***. Taking limit as ***n***→∞, it follows that $v(t) \leq u(t) \leq w(t)$ on ***J***.

Next we obtain the uniqueness of solutions of problem (2.1) in the following

Theorem 3.2 Suppose that

i) $f_i(t, u_1(t), u_2(t))$ is quasi − monotone non − decreasing

ii) $v_o(t)$ and $w_0(t)$ in $C_p(J, \mathrm{R})$ are weakly coupled lower and upper solutions of (2.1) such that $v_0(t) \leq w_0(t)$ on *J=[0,T]*

iii) $f_i(t, u_1(t), u_2(t))$ satisfies Lipschitz condition

$$|f_i(t, u_1(t), u_2(t)) - f_i(t, v_1(t), v_2(t))| \geq M_i |u_i - v_i|, \qquad M_i \geq 0$$

iv) $\lim_{n\to\infty} \|w^n(t) - v^n(t)\| = 0$, where the norm is defined by $\|f\| = \int_0^T |f(s)|\, ds$

Then the solution of problem (2.1) is unique.

Proof. It is sufficient to prove that $v(t) \geq w(t)$. Consider $p_i(t) = w_i(t) - v_i(t)$ we find that

$$D^q p_i(t) = f_i(t, w_1(t), w_2(t)) - f_i(t, v_1(t), v_2(t))$$
$$\leq -M_i p_i(t)$$

and $\quad p_i(0) \leq 0$.

Applying Theorem 2.1, $p_i(t) \leq 0$ implies $v_i(t) \geq w_i(t)$. Combining with $v(t) \leq w(t)$, we obtain $v_i(t) = w_i(t)$. Thus there exists unique solution of problem (2.1) on ***J***.

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