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Some Fixed Point Results in Dislocated Metric Space

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ABSTRACT: *In this article, we establish some common fixed point results for a pair of weakly compatible mappings which generalize and extend the results of A. Amri and D. Moutawakil [1] and W. Sintunavarat and P. Kumam [6] in dislocated metric space.*
Keywords: *d-metric space, common fixed point, weakly compatible maps, cauchy sequence.*
Mathematics Subject Classification: 47H10, 54H25.

I. INTRODUCTION

In 1922, S. Banach proved a fixed point theorem for contraction mapping in metric space. Since then, many generalizations of this theorem have been established in various disciplines. In 1986, S. G. Matthews [5] introduced some concepts of metric domains in the context of domain theory. In 2000, P. Hitzler and A.K. Seda [2] introduced the concept of dislocated topology in which dislocated metric is appeared. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. The purpose of this article is to establish some common fixed point theorems for a pair of weakly compatible mappings with (E. A.) and (CLR) property in dislocated metric space.

II. PRELIMINARIES

We start with the following definitions, lemmas and theorems.

Definition 1 *[2] Let* X *be a non empty set and let* $d : X \times X \to [0, \infty)$ *be a function satisfying the following conditions:*

1. $d(x, y) = d(y, x)$
2. $d(x, y) = d(y, x) = 0$ implies $x = y$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z $\in$ X.

Then d is called dislocated metric (or d-metric) on X and the pair (X, d) is called the dislocated metric space (or d-metric space).

Definition 2 *[2] A sequence* $\{x_n\}$ *in a d-metric space* (X, d) *is called a Cauchy sequence if for given* $\varepsilon > 0$*, there corresponds* $n_0 \in N$ *such that for all* $m, n \geq n_0$ *, we have* $d(x_m, x_n) < \varepsilon$.

Definition 3 *[2] A sequence in d-metric space converges with respect to d (or in d) if there exists* $x \in X$ *such that* $d(x_n, x) \to 0$ *as* $n \to \infty$.

Definition 4 *[2] A d-metric space* (X, d) *is called complete if every Cauchy sequence in it is convergent with respect to* d.

Lemma 1 *[2] Limits in a d-metric space are unique.*

Definition 5 *[4] Let* A *and* S *be mappings from a metric space* (X, d) *into itself. Then,* A *and* S *are said to be weakly compatible if they commute at their coincident point; that is,* $Ax = Sx$ *for some* $x \in X$ *implies* $ASx = SAx$.

Definition 6 *[1] Let A and S be two self mappings defined on a metric space (X,d). We say that the mappings A and S satisfy (E. A.) property if there exists a sequence $\{x_n\} \in X$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$$

for some $u \in X$

Definition 7 *[6] Let A and S be two self mappings defined on a metric space (X,d). We say that the mappings A and S satisfy (CLR_A) property if there exists a sequence $\{x_n\} \in X$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Ax$$

III. MAIN RESULTS

Theorem 1 *Let A and S be two weakly compatible self mappings of a dislocated metric space (X, d) such that*

$$A \text{ and } S \text{ satisfy E.A. the property} \quad (1)$$

$$d(Sx,Sy) < k\max\{d(Ax,Ay), d(Sx,Ax), d(Sy,Ay), d(Sy,Ax), d(Sx,Ay)\} \quad (2)$$

$$\forall x, y \in X,\ k \in [0, \tfrac{1}{2})$$

$$S(X) \subset A(X) \quad (3)$$

If A(X) or S(X) is a complete subspace of X, then A and S have a unique common fixed point.

Proof. Since A and S satisfy E. A. property so there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = u \quad \text{for some } u \in X$$

Let us assume that $A(X)$ is complete, then

$$\lim_{n\to\infty} Ax_n = Ax \quad \text{for some } x \in X$$

Also,

$$\lim_{n\to\infty} Sx_n = Ax$$

Now we sho that $Sx = Ax$. If possible let $Sx \neq Ax$

Now from condition (2) we have,

$$d(Sx_n, Sx) < k\max\{d(Ax_n, Ax), d(Sx_n, Ax_n), d(Sx, Ax), d(Sx, Ax_n), d(Sx_n, Ax)\}$$

Taking limit as $n \to \infty$ we obtain,

$$d(Ax,Sx) < k\max\{0, 0, d(Sx,Ax), d(Sx,Ax), 0\}$$
$$= kd(Ax,Sx)$$

which is a contradiction. Hence, $Ax = Sx$.

Since A and S are weakly compatible, so $ASx = SAx$. Therefore

$$SSx = SAx = ASx = AAx$$

Now we show that Sx is the common fixed point of S and A. Assume that $Sx \neq SSx$. Now by condition (2), we have

$$d(Sx,SSx) < k\max\{d(Ax,ASx), d(Sx,Ax), d(SSx,ASx), d(SSx,Ax), d(Sx,ASx)\}$$
$$= k\max\{d(Sx,SSx), d(Sx,Sx), d(SSx,SSx), d(SSx,Sx), d(Sx,SSx)\}$$
$$= k\max\{d(Sx,SSx), d(Sx,Sx), d(SSx,SSx)\}$$

Since $d(Sx,Sx)\ \&\ \ d(SSx,SSx) \le 2d(Sx,SSx)$

So if $\max\{d(Sx,SSx), d(Sx,Sx), d(SSx,SSx)\} = d(Sx,SSx) \text{ or } d(Sx,Sx) \text{ or } d(SSx,SSx)$,

for all three cases we obtain contradictions. Therefore $d(Sx,SSx) = 0 Sx = SSx$. Hence, Sx is the fixed point of S and $ASx = SSx = Sx$ implies Sx is the common fixed point of the mappings A and S.

The proof is similar when $S(X)$ is supposed to be a complete subspace of X since $S(X) \subset A(X)$.
Uniqueness: Uniqueness follows easily applying condition (2) for supposed any two common fixed points of the mappings A and S.

Now we have the following corollaries.

From Jungck's definition [3], two non compatible self mappings of a metric space (X,d) satisfy E. A. property, so we have the following corollary.

Corrollary 1 *Let A and S be two non-compatible weakly compatible self mappings of a dislocated metric space (X, d) such that*

$$d(Sx,Sy) < k\max\{d(Ax,Ay), d(Sx,Ax), d(Sy,Ay), d(Sy,Ax), d(Sx,Ay)\} \quad (4)$$

$\forall x, y \in X, k \in [0, \frac{1}{2})$

$$S(X) \subset A(X) \quad (5)$$

If A(X) or S(X) is a complete subspace of X, then A and S have a unique common fixed point.

Corrollary 2 *Let A and S be two weakly compatible self mappings of a dislocated metric space (X, d).Suppose, there exists a mapping* $\phi : X \to \mathrm{R}^{+}$ *such that*

$$d(Ax,Sx) < \phi(Ax) - \phi(Sx) \quad \forall x \in X. \quad (6)$$

$$d(Sx,Sy) < k\max\{d(Ax,Ay), d(Sx,Ax), d(Sy,Ay), d(Sy,Ax), d(Sx,Ay)\} \quad (7)$$

$\forall x, y \in X, k \in [0, \frac{1}{2})$

$$S(X) \subset A(X) \quad (8)$$

If A(X) or S(X) is a complete subspace of X, then A and S have a unique common fixed point.

Proof. Choose a sequence $\{x_n\} \in X$ such that $Ax_n = Sx_{n-1}$, then

$$d(Ax_n, Ax_{n+1}) = d(Ax_n, Sx_n) \le \phi(Ax_n) - \phi(Ax_{n+1}) \quad (9)$$

Let us suppose a sequence $\{a_n\}$ defined by $a_n = \phi(Ax_n)$. It is clear that the sequence $\{a_n\}$ is non decreasing and bellowed by 0. This represents that the sequence $\{a_n\}$ is convergent and we have

$$d(Ax_n, Ax_{n+m}) \le a_n - a_{n+m} \quad (10)$$

which implies that the sequence $\{Ax_n\}$ is a Cauchy sequence in $A(X)$. Suppose that $A(X)$ is a complete subspace of X. Then there exists an element $t \in A(x)$ such that $\lim_{n\to\infty} Ax_n = t$. Also we have $\lim_{n\to\infty} Sx_n = t$. This represents that the mappings A and S satisfy the E. A. property. The condition (7) of the corollary and condition (2) of theorem (1) are same. Therefore all the conditions of theorem (1) are satisfied and the conclusion follows immediately.

Theorem 2 *Let A and S be two weakly compatible self mappings of a dislocated metric space (X, d) such that*

$$A \text{ and } S \text{ satisfy E.A. the property} \quad (11)$$

$$d(Sx,Sy) < k\{d(Ax,Ay) + d(Sx,Ax) + d(Sy,Ay) + d(Sy,Ax) + d(Sx,Ay)\} \quad (12)$$

$\forall x, y \in X, k \in [0, \frac{1}{7})$

$$S(X) \subset A(X) \quad (13)$$

If A(X) or S(X) is a complete subspace of X, then A and S have a unique common fixed point.

Proof. Since A and S satisfy E. A. property so there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = u \quad for\, some\, u \in X$$

Let us assume that $A(X)$ is complete, then

$$\lim_{n\to\infty} Ax_n = Ax \quad for\, some\, x \in X$$

Also,

$$\lim_{n\to\infty} Sx_n = Ax$$

Now we sho that $Sx = Ax$. If possible let $Sx \neq Ax$

Now from condition (12) we have,

$$d(Sx_n, Sx) < k\{d(Ax_n, Ax) + d(Sx_n, Ax_n) + d(Sx, Ax) + d(Sx, Ax_n) + d(Sx_n, Ax)\}$$

Taking limit as $n \to \infty$ we obtain,

$$d(Ax, Sx) < k\{0 + 0 + d(Sx, Ax) + d(Sx, Ax) + 0\}$$
$$= 2kd(Ax, Sx)$$

which is a contradiction. Hence, $Ax = Sx$.

Since A and S are weakly compatible, so $ASx = SAx$. Therefore

$$SSx = SAx = ASx = AAx$$

Now we show that Sx is the common fixed point of S and A. Assume that $Sx \neq SSx$. Now by condition (12), we have

$$d(Sx, SSx) < k\{d(Ax, ASx) + d(Sx, Ax) + d(SSx, ASx) + d(SSx, Ax) + d(Sx, ASx)\}$$
$$= k\{d(Sx, SSx) + d(Sx, Sx) + d(SSx, SSx) + d(SSx, Sx) + d(Sx, SSx)\}$$
$$= k\{3d(Sx, SSx) + d(Sx, Sx) + d(SSx, SSx)\}$$
$$\leq 7kd(Sx, SSx)$$

which is a contradictions. Therefore $d(Sx, SSx) = 0 Sx = SSx$. Hence, Sx is the fixed point of S. and $ASx = SSx = Sx$ implies Sx is the common fixed point of the mappings A and S.

The proof is similar when $S(X)$ is supposed to be a complete subspace of X since $S(X) \subset A(X)$.

Uniqueness: Uniqueness follows easily applying condition (12) for supposed any two common fixed points of the mappings A and S.

Now we have the following corollaries.

From Jungck's definition [3], two non compatible self mappings of a metric space (X, d) satisfy E. A. property, so we have the following corollary.

Corollary 3 *Let A and S be two non-compatible weakly compatible self mappings of a dislocated metric space (X, d) such that*

$$d(Sx, Sy) < k\{d(Ax, Ay) + d(Sx, Ax) + d(Sy, Ay) + d(Sy, Ax) + d(Sx, Ay)\}$$

$\forall x, y \in X, k \in [0, \frac{1}{7})$

$$S(X) \subset A(X)$$

If A(X) or S(X) is a complete subspace of X, then A and S have a unique common fixed point.

Theorem 3 *Let A and S be two weakly compatible self mappings of a dislocated metric space (X, d) such that*

$$A \text{ and } S \text{ satisfy } CLR_A \text{ or } CLR_S \text{ property} \tag{14}$$

$$d(Sx, Sy) < k\max\{d(Ax, Ay), d(Sx, Ax), d(Sy, Ay), d(Sy, Ax), d(Sx, Ay)\} \tag{15}$$

$\forall x, y \in X, k \in [0, \frac{1}{2})$ then the mappings A and S have a unique common fixed point.

Proof. Since A and S satisfy CLR_A property so there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = Ax \quad for\, some\; x \in X$$

We show that $Ax = Sx$

Now from condition (15) we have,

$$d(Sx_n, Sx) < kmax\{d(Ax_n, Ax), d(Sx_n, Ax_n), d(Sx, Ax), d(Sx, Ax_n), d(Sx_n, Ax)\}$$

Taking limit as $n \to \infty$ we obtain,

$$d(Ax, Sx) < kmax\{0, 0, d(Sx, Ax), d(Sx, Ax), 0\}$$
$$= kd(Ax, Sx)$$

which is a contradiction. Hence, $Ax = Sx$.

Since A and S are weakly compatible, so $ASx = SAx$. Therefore

$$SSx = SAx = ASx = AAx$$

Now we show that Sx is the common fixed point of S and A. Assume that $Sx \neq SSx$. Now by condition (15), we have

$$d(Sx, SSx) < kmax\{d(Ax, ASx), d(Sx, Ax), d(SSx, ASx), d(SSx, Ax), d(Sx, ASx)\}$$
$$= kmax\{d(Sx, SSx), d(Sx, Sx), d(SSx, SSx), d(SSx, Sx), d(Sx, SSx)\}$$
$$= kmax\{d(Sx, SSx), d(Sx, Sx), d(SSx, SSx)\}$$

Since $d(Sx, Sx)$ & $d(SSx, SSx) \leq 2d(Sx, SSx)$

So if $max\{d(Sx, SSx), d(Sx, Sx), d(SSx, SSx)\} = d(Sx, SSx)\, or\, d(Sx, Sx)\; or\; d(SSx, SSx)$,

for all three cases we obtain contradictions. Therefore $d(Sx, SSx) = 0 Sx = SSx$. Hence, Sx is the fixed point of S. and $ASx = SAx = SSx = Sx$ implies Sx is the common fixed point of the mappings A and S.

The proof is similar for the case when the mappings satisfy CLR_S property.

Uniqueness: Uniqueness follows easily applying condition (15) for supposed any two common fixed points of the mappings A and S.

Theorem 4 *Let A and S be two weakly compatible self mappings of a dislocated metric space (X, d) such that*

$$A \text{ and } S \text{ satisfy } CLR_A \text{ or } CLR_S \text{ property} \qquad (16)$$

$$d(Sx, Sy) < k\{d(Ax, Ay) + d(Sx, Ax) + d(Sy, Ay) + d(Sy, Ax) + d(Sx, Ay)\} \qquad (17)$$

$\forall x, y \in X$, $k \in [0, \frac{1}{7})$ then the mappings A and S have a unique common fixed point.

Proof. Since A and S satisfy CLR_A property so there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = Ax \quad for\, some\; x \in X$$

We show that $Ax = Sx$ Now from condition (17) we have,

$$d(Sx_n, Sx) < k\{d(Ax_n, Ax) + d(Sx_n, Ax_n) + d(Sx, Ax) + d(Sx, Ax_n) + d(Sx_n, Ax)\}$$

Taking limit as $n \to \infty$ we obtain,

$$d(Ax, Sx) < k\{0 + 0 + d(Sx, Ax) + d(Sx, Ax) + 0\}$$
$$= 2kd(Ax, Sx)$$

which is a contradiction. Hence, $Ax = Sx$.

Since A and S are weakly compatible, so $ASx = SAx$. Therefore

$$SSx = SAx = ASx = AAx$$

Now we show that Sx is the common fixed point of S and A. Assume that $Sx \neq SSx$. Now by condition (17), we have

$$
\begin{aligned}
d(Sx,SSx) &< k\{d(Ax,ASx)+d(Sx,Ax)+d(SSx,ASx)+d(SSx,Ax)+d(Sx,ASx)\} \\
&= k\{d(Sx,SSx)+d(Sx,Sx)+d(SSx,SSx)+d(SSx,Sx)+d(Sx,SSx)\} \\
&= k\{3d(Sx,SSx)+d(Sx,Sx)+d(SSx,SSx)\} \\
&\le 7kd(Sx,SSx)
\end{aligned}
$$

which is a contradictions. Therefore $d(Sx,SSx)=0Sx=SSx$. Hence, Sx is the fixed point of S. and $ASx=SAx=SSx=Sx$ implies Sx is the common fixed point of the mappings A and S.
The proof is similar for the case when the mappings satisfy CLR_S property.

Uniqueness: Uniqueness follows easily applying condition (17) for supposed any two common fixed points of the mappings A and S.

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