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Using Fourth-Order Runge-Kutta Method to Solve Lü Chaotic System

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ABSTRACT: *Various results in the previous literature exist that indicate that all computed solutions to chaotic dynamical systems are time step dependent. These problems can be solved by using multiple methods. This paper deals with an explicit MATHEMATICA algorithm for the implementation of Runge-Kutta method of orders 4 (RK4) to solve the Lü chaotic system. Numerical comparisons are made between the Runge-Kutta of fourth-order and the Euler's method. Comparisons were also done between the (RK4) methods but with different time steps. It has been observed that the accuracy of (RK4) solutions can be increased by lessening the time step. And shows that (RK4) method successfully to solve the Lü system. It has been determine accuracy of method using symmetrical times.*

Keywords: *Lü chaotic system, Fourth-order Runge-Kutta method , dynamical systems and Euler's method.*

I. INTRODUCTION

During 1990, Ott, Grebogi and Yorke offered the (OGY) method to control chaos [1] . After their founding work,chaotic control has become a focus in nonlinear problems and a lot of work has been done in the field [2]-[4]. Currently, many methods have been proposed to control chaos [4] [5]. Most of scientific problems and natural phenomena can be modeled by chaotic systems of ordinary differential equations (ODEs). Not all chaotic systems have analytical solutions. This is due to their difficulties. Therefore, the numerical methods can be used to obtain the approximation of solutions of the problems. Some numerical methods that can be used to solve the systems are; Euler's method, midpoint method, Heun's method and Runge-Kutta method of different orders. Newly, Hairer *et al.* [6] have used the Euler's method to solve the chaotic system. They used this method because it is one of the simplest approaches to obtain the numerical solution of a differential equation. An algorithm for Euler's method is used to obtain an approximation for the initial-conditions problem and was employed to Lü's chaotic system [7]. They used the FORTRAN software to solve this system and MATHMATICA to plot the solutions and the results are given for different number of iterations. Although the results obtained is the same butterfly effect, but however, this method is not an efficient method and seldom used because of its less accuracy [8]. In this paper, has been interested to test the system [7]. We choose the RK4 because it can obtain greater accuracy and does not need the calculation of higher derivatives [8]. Moreover, RK4 has been widely and commonly used for simulating the solution of chaotic systems [9, 10, 11, 12] .We want to prove whether this method successfully can solve the Lü system or not. The organization of this paper is in the following manner. In Section 2 we give some introduction for a new chaotic system that is Lü system. The definition of fourth-order Runge-Kutta method (RK4) will be defined in Section 3 while in Section 4 we show the algorithm to compute the RK4. Section 5 is the numerical results and discussion.

II. THE LÜ CHAOTIC SYSTEM

The Lü system is defined by them followings equations:

$$\boldsymbol{u} = \begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz \end{cases} \qquad (1)$$

Where a, b, c are real constants. When a=36 , b =3 , c = 20, Lü system has a typically critical chaotic attractor with Lyapunov exponents L_1= 1.5046, L_2= 0, L_3 = -22.5044 and Lyapunov dimension d_L= 2.0669 [7]. Figure 1 to Figure 4 shows the attractor Zhou system from different views.

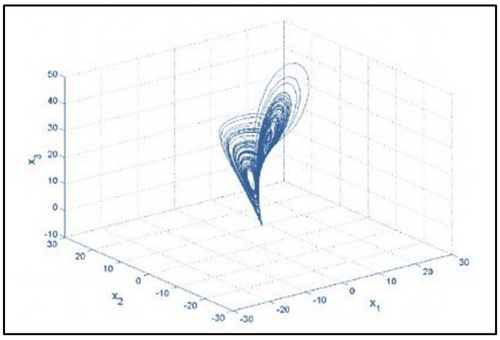

Figure 1. The chaotic attractor of Lü system.

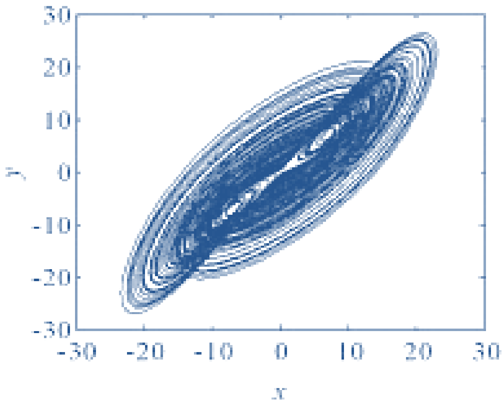

Figure 2: *x-y* phase plane of Lü's attractor

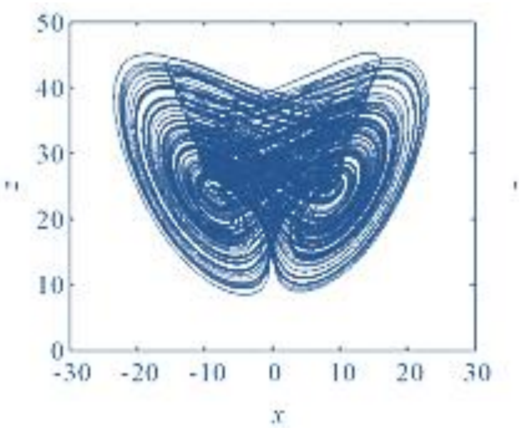

Figure 3: *x-z* phase plane of Lü's attractor

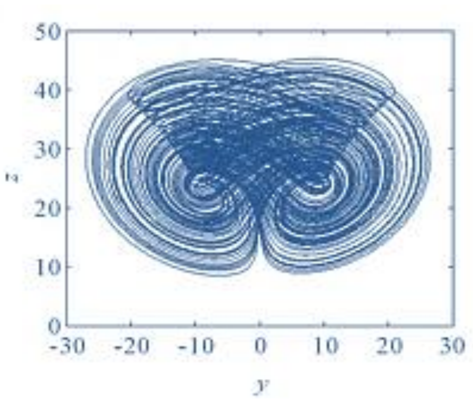

Figure 4: *y-z* phase plane of Lü's attractor

III. FOURTH-ORDER RUNGE-KUTTA METHOD (RK4)

There exists some different orders of Runge-Kutta methods, but all of them can be cast in the following general form.

$$y_{i+1} = y_i + \phi(t_i, y_i, h)h \qquad (2)$$

Where $\phi(t_i, y_i, h)$ is named an increment function, which is interpreted as the representative slope over interval. The estimate slope ϕ is used to extrapolate from an old value y_i to a new value y_{i+1} over a distance h. This is called an explicit method. The general form of this increment function is:

$$\phi = a_1k_1 + a_2k_2 + \ldots + a_nk_n \qquad (3)$$

Where the a's are real constants and the k's are:

$k_1 = f(t_i, y_i)$

$k_2 = f(t_i + p_1h, y_i + q_{11}k_1h)$

$k_3 = f(t_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h)$

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$k_n = f(t_i + p_{n-1}h, y_i + q_{n-1,1}k_1h + q_{n-1,2}k_2h + \ldots + q_{n-1,n-1}k_{n-1})$

Where p's and q's are constants [13].

To solve (ODEs) problem, an initial value problem (IVP) of the first order differential equation consider as :

$$y' = f(t, y), \; a \le t \le b, \; y(a) = \alpha \qquad (4)$$

The solution of this IVP by using the classical RK4 is given by:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (5)$$

Where:

$$k_1 = hf(t_i, y_i),$$
$$k_2 = hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right),$$
$$k_3 = hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right),$$
$$k_4 = hf(t_i + h, y_i + k_3),$$

This explicit Runge-Kutta method of order four (RK4) requires four evaluations of function [14]. We will use this classical RK4 method to solve the Lü chaotic system which will be explained in the next section.

IV. THE ALGORITHM OF RK4

Follow is the algorithm to calculate the (RK4) as stated in Burden and Faires [15]. We will apply this algorithm to solve the Lü chaotic system in order to find the values of x, y and z with the initial conditions (0, 4, 1).

To approximate the solution of the (IVP) in (4) at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT: endpoints a,b; integer N; initial value α

OUTPUT: approximation w to y at the $(N + 1)$ values of t

Step 1: set $\Delta t = (b-a)/N$;

$t = a;$

$w = a;$

OUTPUT (t,w)

Step 2: for $i = 1,2,\ldots, N$ do steps 3-5.

Step 3: set

$$k_1 = hf(t, w);$$
$$k_2 = hf\left(t + \frac{h}{2}, w + \frac{1}{2}k_1\right);$$
$$k_3 = hf\left(t + \frac{h}{2}, w + \frac{1}{2}k_2\right);$$
$$k_4 = hf(t + h, w + k_3).$$

Step 4: set

$$w = w + (k_1 + k_2 + k_3 + k_4)/6,$$

(*Compute w_i*)
T=a+ih (compute t_i)
Step 5: OUTPUT (t,w)
Step 6: STOP.
The algorithm above has been applied to solve the Lü system by using MATHEMATICA program to plot the solutions.

V. RESULTS AND DISCUSSIONS

Hairer *et al*. [6] have used the Euler's FORTRAN software to solve the system and MATHEMATICA to plot the solutions of *x*, *y* and *z*. In this paper, we first make the comparison between the Euler's methods with different solution of equation (1) for different time steps. we compare the accuracy of the (RK4) method with the Euler's method on the chosen time step $\Delta t=10^{-8}$. The absolute values were used to determine the performance of (RK4) against the Euler's method. In Table 1, we first find the error between the RK4 method ($\Delta t = 0.01$) and Euler ($\Delta t=10^{-8}$). We could see clearly that the maximum error is 8.7113937.

So, we can conclude that the accuracy of (RK4) solutions can be increased by decreasing the time step. To solve the three-dimensional system of Lü chaotic system, we use the Maple program to run the (RK4) in order to produce the values of *x*, *y* and *z* when the value of time increased. Then these values will be linked to MATHEMATICA program to plot the solutions. The result is shown below in Figure 5 when $0\leq t\leq 10$ Here, we choose $\Delta t = 0.001$ to solve the Lü system. With the time steps of 0.001, this means that there are 6000 values of *x*, *y* and *z*. Notice that this figure has only one part of butterfly wings. This is due to the lower numbers of iterations used which are 6000. Next, we show the effect of different ranges of *t* to Lü's attractor with the same time steps; 0.001. The more the iterations used, the more the attractor become complete. By using different number of iterations, we can see how the attractor is designed and moves. The result is shown below in Figure 6 to Figure 8.

Table 1: Differences between RK4 and Euler solutions for $t\in[0,100]$

	$\Delta = RK4_{0.01}$ - $EULER_{10^{-8}}$			$\Delta = RK4_{0.001}$ - $EULER_{10^{-8}}$		
Time	**x**	**y**	**z**	**x**	**y**	**Z**
10	**6.7016406**	**4.1285734**	**0.7021884**	**6.7016415**	**4.1285742**	**0.7021891**
20	**2.2680077**	**4.0650627**	**5.0595822**	**2.2680068**	**4.0650651**	**5.0595873**
30	**0.2387785**	**2.7060230**	**4.7138815**	**0.2387731**	**2.7060303**	**4.7138816**
40	**9.1531329**	**7.1423708**	**3.1561032**	**9.1531488**	**7.1423711**	**3.1561505**
50	**0.2897943**	**4.2923150**	**5.7520798**	**0.2898451**	**4.2923377**	**5.7522011**
60	**1.2252843**	**2.5261763**	**4.5136376**	**1.2253142**	**2.5262133**	**4.5135830**
70	**7.8610576**	**8.3377190**	**1.1478526**	**7.8610183**	**8.3379224**	**1.1473481**
80	**2.0993062**	**0.5678702**	**0.2870140**	**2.0993051**	**0.5680154**	**0.2705307**
90	**0.5229035**	**3.7653424**	**2.4082413**	**0.5231548**	**3.7656263**	**2.4075923**
100	**8.7113937**	**6.4285621**	**1.8642130**	**8.7039811**	**6.4274910**	**1.8810160**

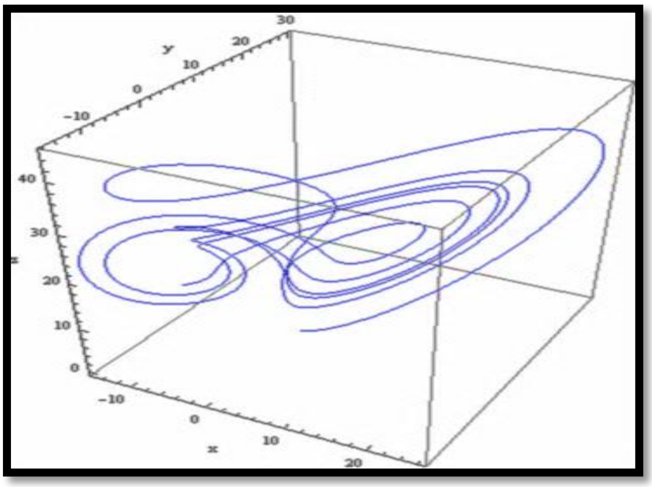

Figure 5: Lü's attractor when $0\leq t\leq 10$

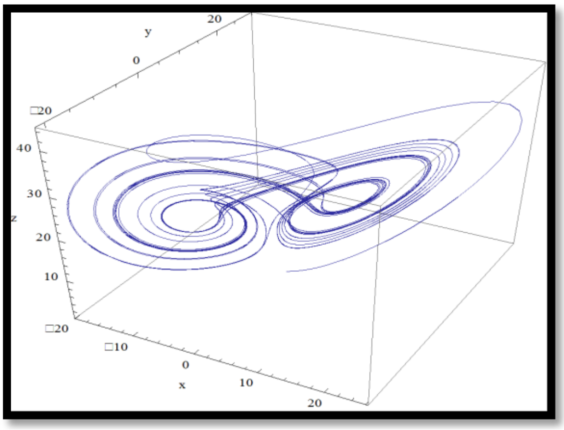

Figure 6: Lü's attractor when $0 \le t \le 20$

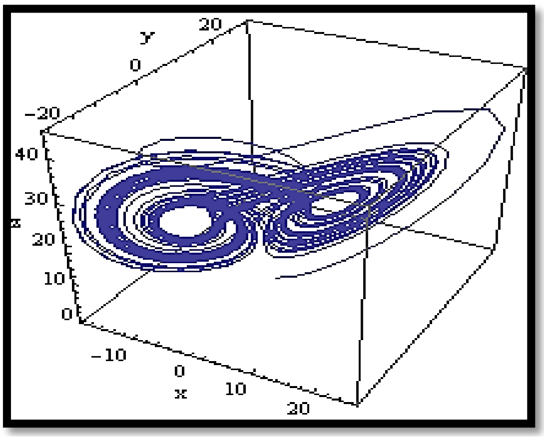

Figure 7: Lü's attractor when $0 \le t \le 50$

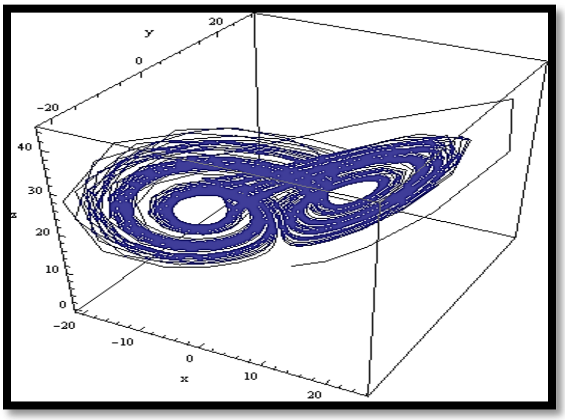

Figure 8: Lü's attractor when $0 \le t \le 100$

VI. CONCLUSIONS

In this paper, we calculated time step independent solutions up to t = 100 with two different numerical methods. This paper shows that the (RK4) method successfully to solve the Lü chaotic system [7]. This method is used because it (RK4) can obtain greater accuracy and does not need the calculation of higher derivatives. From the previous research by Hairer *et al*. [6], the use of Euler's method can also solve the chaotic system, but however, it is less accuracy compared to (RK4) method. Numerical comparisons have been made between the Runge-Kutta of order four (RK4) and the Euler's method for different time steps. It has been observed that the accuracy of (RK4) solutions can be increased by lessening the time step. An algorithm for (RK4) method is used to solve the initial-value problem for ordinary differential equation of the Lü chaotic system. This method yields the values of *x*, *y* and *z*. The results are given for different ranges of time. And using different number of repetitions, we can see how the attractor is designed and changes.

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