American Journal of Engineering Research (AJER)
e-ISSN: 2320-0847 p-ISSN : 2320-0936
Volume-02, Issue-02, pp-01-06
www.ajer.us

Research Paper Open Access

Structural Properties of length biased Beta distribution of first kind

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Abstract: In this paper, a new class of Length-biased beta distribution of first kind is introduced. A Length-biased beta distribution of first kind is a particular case of the weighted beta distribution of first kind, taking the weights as the variate values has been defined. The characterizing properties of the model are derived. The estimates of parameters of Length-biased beta distribution of first kind are obtained by using method of moments. Also, a test for detecting the length-baisedness is conducted.

Key Words: *Beta distribution of first kind, Beta function, Length-biased beta distribution of first kind, structural properties, moment estimator, likelihood ratio test.*

I. INTRODUCTION

Beta distributions are very versatile and a variety of uncertainties can be usefully modelled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modelled by finite range distributions, see for example Barlow and Proschan [1].

A continuous random variable X is said to have a beta distribution of first kind with parameters a and b if probability density function (pdf) is:

$$f(x;a,b) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \quad (1.1)$$

$for\, 0 < x < 1, a > 0\, and\ b > 0, where$

$\beta(a,b) = \int_o^1 t^{a-1}(1-t)^{b-1}dt$ denotes the beta function.

Many generalizations of beta distributions involving algebraic and exponential functions have been proposed in the literature; see in Johnson et al [2] and Gupta and NadarSajah [3] for detailed accounts.

II. DERIVATION OF LENGTH-BIASED BETA DISTRIBUTION OF FIRST KIND:

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to measure of the unit size, resulting distribution is called size-biased. Size biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher [4] to model ascertainment bias, these were later formalized in a unifying theory by Rao [5]. These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non –experimental, non –replicated and non –random categories. . Van Deusen [6] arrived at size biased distribution theory independently and applied it to fitting distributions of diameter at breast height (DBH) data arising from horizontal point sampling (HPS) (Grosenbaugh) inventories. Subsequently, Lappi and Bailey [7] used weighted distributions to analyse HPS diameter increment data. Dennis and Patil [8] used stochastic differential equations to arrive at a weighted gamma distribution as the stationary probability density function (PDF) for the stochastic population model with predation effects. Gove [9] reviewed some of the more recent results on size-biased distributions pertaining to parameter estimation in forestry. Mir [10] also discussed some of the discrete size-biased distributions.

If the random variable X has distribution $f(x;\theta)$, with unknown parameter θ, then the corresponding size – biased distribution is of the form

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu_c'} \qquad (2.1)$$

$$\mu_c' = \int x^c f(x;\theta)dx \qquad \text{For continuous series} \qquad (2.2)$$

$$\mu_c' = \sum_{i=1}^{n} x^c f(x;\theta)dx \qquad \text{For discrete series.}$$

When c = 1 and 2, we get the size –biased and area biased - distributions respectively.

The probability distribution of Beta distribution of first kind is:

$$f(x;a,b) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}$$

$$for\, 0 < x < 1, a > 0\, and\ b > 0, where$$

A length biased beta distribution of first kind (LBBD1) is obtained by applying the weights x^c, where c =1 to the weighted beta distribution.

$$\mu_1' = \int_0^1 x\, f(x;a,b)dx = \frac{a}{a+b}$$

$$\int_0^1 x \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}dx = \frac{a}{a+b}$$

$$\int_0^1 \frac{1}{\beta(a,b)} \frac{a+b}{a} x^{a}(1-x)^{b-1}dx = 1$$

$$f(x;a+1,b) = \frac{1}{\beta(a+1,b)} x^{a}(1-x)^{b-1} \qquad (2.3)$$

Where $f(x;a,b+1)$ represents a probability density function. This gives the length –biased beta distribution of first kind (LBBD1) as:

$$f(x;a+1,b) = \frac{1}{\beta(a+1,b)} x^{a}(1-x)^{b-1}; a \geq 0, b > 0 \qquad (2.4)$$

$$= 0; otherwise \qquad 0 < x < 1$$

Where $\beta(a,b) = \int_o^1 t^{a-1}(1-t)^{b-1}dt$

III. STRUCTURAL PROPERTIES OF LENGTH BIASED BETA DISTRIBUTION OF FIRST KIND:

The rth moment of Length biased beta distribution of first kind (2.4) about origin is obtained as:

$$\mu_r' = \int_0^1 x^r f(x;a+1,b)dx$$

$$\mu_r' = \int_0^1 x^r \frac{1}{\beta(a+1,b)} x^{a}(1-x)^{b-1}dx$$

On solving the above equation, we get

$$\mu_r' = \int_0^1 \frac{1}{\beta(a+1,b)} x^{a+r}(1-x)^{b-1}dx$$

$$\mu_r' = \frac{1}{\beta(a+1,b)}\beta(a+r+1,b) \qquad (3.1)$$

3.1 Mean of length biased Beta Distribution of first kind.

Using the equation (3.1), the mean of the LBBD1 is given by

$$\mu_1' = \frac{1}{\beta(a+1,b)}\beta(a+2,b)$$

$$\mu_1' = \frac{\Gamma a+b+1}{\Gamma a+1\,\Gamma b}\,\frac{\Gamma a+2\ \Gamma b}{\Gamma a+b+2}$$

$$\mu_1' = \frac{a+1}{a+b+1} \qquad (3.2)$$

3.2 Second moments of length biased Beta Distribution of first kind.

Using the equation (3.1), the second moments of the LBBD1 is given by

$$\mu_2' = \frac{1}{\beta(a+1,b)}\beta(a+3,b)$$

$$\mu_2' = \frac{\Gamma a+b+1}{\Gamma a+1\,\Gamma b}\,\frac{\Gamma a+3\ \Gamma b}{\Gamma a+b+3}$$

$$\mu_2' = \frac{(a+1)(a+2)}{(a+b+1)(a+b+2)} \qquad (3.3)$$

3.3 Variance of length biased Beta Distribution of first kind.

Using the equations (3.2) and (3.3), the variance of the LBBD1 is given by

$$\mu_2 = \frac{(a+1)b}{(a+b+1)^2(a+b+2)} \qquad (3.4)$$

3.4 Third and fourth moments of length biased Beta Distribution of first kind.

Using the equation (3.1), the third and fourth moments of the LBBD1 is given by

$$\mu_3' = \frac{1}{\beta(a+1,b)}\beta(a+4,b)$$

$$\mu_3' = \frac{\Gamma a+b+1}{\Gamma a+1\,\Gamma b}\,\frac{\Gamma a+4\ \Gamma b}{\Gamma a+b+4}$$

On solving the above equation, we get

$$\mu_3' = \frac{(a+1)(a+2)(a+3)}{(a+b+1)(a+b+2)(a+b+3)} \qquad (3.5)$$

$$.\ \mu_4' = \frac{1}{\beta(a+1,b)}\beta(a+5,b)$$

$$\mu_4' = \frac{\Gamma a+b+1}{\Gamma a+1\,\Gamma b}\,\frac{\Gamma a+5\,\Gamma b}{\Gamma a+b+5}$$

On solving the above equation, we get

$$\mu_4' = \frac{(a+1)(a+2)(a+3)(a+4)}{(a+b+1)(a+b+2)(a+b+3)(a+b+4)} \tag{3.6}$$

3.5 The coefficient of variation of length biased beta distribution is given as:

$$CV = \left[\frac{b}{(a+1)(a+b+2)}\right]^{\frac{1}{2}} \tag{3.7}$$

3.6 The coefficient of skewness of length biased beta distribution is given as

$$CS = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'}{\sigma^3}$$

3.7 The coefficient of kurtosis of length biased beta distribution is given as

$$CS = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'}{\sigma^3}$$

Where, the first four moments about origin is:

$$\mu_1' = \frac{a+1}{a+b+1}$$

$$\mu_2' = \frac{(a+1)(a+2)}{(a+b+1)(a+b+2)}$$

$$\mu_3' = \frac{(a+1)(a+2)(a+3)}{(a+b+1)(a+b+2)(a+b+3)}$$

$$\mu_4' = \frac{(a+1)(a+2)(a+3)(a+4)}{(a+b+1)(a+b+2)(a+b+3)(a+b+4)}$$

3.8 Harmonic mean of length biased Beta distribution of first kind.

The harmonic mean (H) is given as:

$$\frac{1}{H} = \int_0^1 \frac{1}{x} f(x; a+1, b)dx$$

$$\frac{1}{H} = \int_0^1 \frac{1}{x} \frac{1}{\beta(a+1,b)} x^a (1-x)^{b-1} dx$$

$$\frac{1}{H} = \int_0^1 \frac{1}{\beta(a+1,b)} x^{a-1} (1-x)^{b-1} dx$$

$$\frac{1}{H} = \frac{1}{\beta(a+1,b)} \beta(a,b)$$

$$\frac{1}{H} = \frac{\Gamma a+b+1}{\Gamma a+1 \, \Gamma b} \frac{\Gamma a \, \Gamma b}{\Gamma a+b}$$

$$\frac{1}{H} = \frac{a+b}{a}$$

$$H = \frac{a}{a+b} \tag{3.8}$$

3.9 The mode of length biased beta distribution of first kind is given as:

The probability distribution of Length- biased Beta distribution of first kind is:

$$f(x;a+1,b)=\frac{1}{\beta(a+1,b)}x^{a}(1-x)^{b-1};a\geq 0,b>0$$
$$=0; otherwise \qquad 0<x<1$$

In order to discuss monotonicity of length biased beta distribution of first kind. We take the logarithm of its pdf:

$$\ln(f(x:a+1,b))=\ln C+a\ln x+(b-1)\ln(1-x)$$

Where C is a constant. Note that

$$\frac{\partial \ln f(x;a+1,b)}{\partial y}=\frac{a}{x}-\frac{b-1}{1-x}$$

Where $x>0,a\geq 0,b>0$. It follows that

$$\frac{\partial \ln f(x;a+1,b)}{\partial y}>0\Leftrightarrow x<\frac{a}{a+b-1}$$
$$\frac{\partial \ln f(x;a+1,b)}{\partial y}=0\Leftrightarrow x=\frac{a}{a+b-1}$$
$$\frac{\partial \ln f(x;a+1,b)}{\partial y}<0\Leftrightarrow x>\frac{a}{a+b-1}$$

Therefore, the mode of length biased beta distribution of first kind is:

$$x_0=\frac{a}{a+b-1} \qquad (3.9)$$

IV. ESTIMATION OF PARAMETERS OF LENGTH-BIASED BETA DISTRIBUTION OF FIRST KIND:

In this method of moments replacing the population mean and variance by the corresponding sample mean and variance, we have:

$$\mu_1'=\bar{x}$$
$$\frac{a+1}{a+b+1}=\bar{x}$$
$$\hat{b}=\frac{-(a+1)(\bar{x}-1)}{\bar{x}} \qquad (4.1)$$

Also, $\mu_2=S^2$

$$\frac{(a+1)b}{(a+b+1)^2(a+b+2)}=S^2$$
$$\hat{a}=\frac{\bar{x}^2(1-\bar{x})-S^2(1+\bar{x})}{S^2} \qquad (4.2)$$

Substitute the value of $\hat{a}$ in the above equation; we can get the estimated value of parameter b.

$$\hat{b}=\frac{(\bar{x}^2+S^2\bar{x}^2)(\bar{x}-1)}{\bar{x}S^2} \qquad (4.3)$$

V. TEST FOR LENGTH-BIASED BETA DISTRIBUTION OF FIRST KIND.

Let X_1, X_2, X_3… Xn be random samples can be drawn from beta distribution of first kind or size-biased beta distribution of first kind. We test the hypothesis $H_o: f(x)=f(x,a,b)\, againest\, H_1: f(x)=f_s^*(x;a,b)$.

To test whether the random sample of size n comes from the beta distribution of first kind or Length-biased beta distribution of first kind the following test statistic is used.

$$\Delta=\frac{L_1}{L_0}=\prod_{i=1}^{n}\frac{f^*{}_s(x;a+1,b)}{f(x;a,b)}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\dfrac{\Gamma(a+b+1)x^a(1-x)^{b-1}}{\Gamma(a+1)\Gamma(b)}}{\dfrac{\Gamma(a+b)x^{a-1}(1-x)^{b-1}}{\Gamma(a)\Gamma(b)}}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\dfrac{\Gamma(a+b+1)x^a}{\Gamma(a+1)}}{\dfrac{\Gamma(a+b)x^{a-1}}{\Gamma(a)}}$$

$$\Delta = \left[\frac{\Gamma(a)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(a+b)}\right]^n \prod_{i=1}^{n} x_i \qquad (5.1)$$

We reject the null hypothesis.

$$\left[\frac{\Gamma(a)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(a+b)}\right]^n \prod_{i=1}^{n} x_i > k$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where\, k^* = k\left[\frac{\Gamma(a+1)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b)}\right]^n > 0 \qquad (5.2)$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution.

VI. CONCLUSION

This paper deals with the length biased form of the weighted Beta distribution of first kind (WBD1) named as a new class of length biased Beta distribution of first kind (LBBD1). A length biased beta distribution of first kind; a particular case of the weighted Beta distribution of first kind, taking the weights as the variate values has been defined. The structural properties of length biased Beta distribution of first kind (LBBD1) including moments, variance, mode and harmonic mean, coefficient of variation, skewness and kurtosis. The estimates of the parameters of length biased Beta distribution of first kind (LBBD1) are obtained by employing the method of moments. Also, a test for detecting the length-baisedness is conducted.

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