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On Generalized Hardy's Inequality and Series of Fractal Measures

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We show Hardy's inequality and the subsequent improvement by McGehee, Pigno, andSmith are generalized from the positive integers to sets of dimension 0, dimension1, and in between. The asymptotic estimate obtained for the Fourier transform of the series of fractal measures is much in the spirit of recent work by Strichartz,Steve Hudson and Mark Leckband .

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I. INTRODUCTION

An interesting problem in Fourier analysis is to extend the classical inequalities of the Fourier transform , or what Hardy and Littlewood refer to as the theory of Fourier constant [5] , to tempered distributions that correspond to lower -dimensional sets.Prticularly important theorems are the L^1 inequality known as Hardy's inequality with the McGehee-Pigono- Smith (henceforth M . P .S .)generalization .[7], the Plancherel theorem for L^2 and Payley's theorem with the Pitt-Stein [generalization for$L^{\varepsilon+1}$, $0 < \varepsilon \leq 1$.Extensions of the Plancherel theorem for measures supported on manifolds inR^n have been established by Agmon and Hormander[1], and more recently by Strichartz [9] for measures on R^n of dimension$-1 < \varepsilon < n-1$,$1+\varepsilon$ not necessarily an integer . We show ,in the same methodology, an application on the paper ofSteve Hudson and Mark Leckband[11], they proved a generalized Hardy inequality (henceforth g.h.i) for fractal measures on R^1 of dimension $(1-\varepsilon)$, $0 < \varepsilon \leq 1$ .This result includes the M.P.S. version as the periodic case for$\varepsilon = 1$.Each of the result above for$\varepsilon < n-1$ involves a limit on the Fourier transform side and provides information in the form of an asymptotic growth estimate for the transform

Some regularity will be required of the support of the fractal measure classically, Hardy's inequality and the M.P.S version hold only for measure. Supported on a well ordered set of integers, which means the transform of the measure is in $H^1$of the unit circle , at least up to a multiplicative factor of $e^{in\pi}$. The well – known inequalities above, in which $\varepsilon > 0$, are rearrangement – invariant , while Hardy's inequality is not . This implies that the nature of the support of the measure when $\varepsilon = 1$ or $\varepsilon = 0$ is for more important in Hardy's inequality than in the others. Likewise,when $0 < \varepsilon < 1$, it is natural to expect the support of the measure to play a greater role in $g.h.i$, then in the other inequalities . This point may be clarified by the last result of the paper , an extension of Paley's theorem for $0-$ dimensional measures , this is a $\varepsilon > 0$ analogue of $g.h.i$ in which the support is quite arbitrary (see [11]) .

Aseries of fractal measures means a measure ν^rsupported on a set $E_r \subset R^1$. That is $\mu_{1-\varepsilon}$ -measurable, where $d\mu_{1-\varepsilon}$ is $(1-\varepsilon)$ -dimensional Hausdorff measures and$0 \leq \varepsilon \leq \mathbf{1}$. Certain classes of such measure will be studied including measure supported a self – similar sets such as the cantor set .

It assumed thatν^r is finite, so it is a tempered distribution with a Fourier transform locally in $L^1(R)$. It is also assumed thatν^r is either positive , or is of the form$f^r d\mu_{1-\varepsilon}$. In the latter case , let $E_r = \{x: f^r(x) \neq 0\}$, which we will refer to as supp f^r.Definethe series

$$\sum_{r\geq 1}\sigma_{1-\varepsilon}^{r}(E_r,x)=\sum_{r\geq 1}\mu_{1-\varepsilon}\big(E_r\cap(-\infty,x)\big). \tag{1.1}$$

Consider the followingseries of generalized Hardy inequality (see [11])$g.h.i$for $0\leq\varepsilon\leq 1$

$$\int_{-\infty}^{\infty}\sum_{r\geq 1}\frac{|f^r(x)|d\mu_{1-\varepsilon}(x)}{\sigma_{1-\varepsilon}^{r}(E_r\ ,x)}\leq \liminf_{L\to\infty} L^{\varepsilon}\sum_{r\geq 1}C^r\int_{-\infty}^{\infty}f^r\, d\widehat{\mu_{1-\varepsilon}(x)}dx \tag{1.2}$$

where C^r is a constant that may depend on E_r but not f^r .This inequality does not hold for general fractal measures . The collective statement of Theorem 1 and 3 is that$g.h.i$ holds wherever E_r is $(1-\varepsilon)$- coherent, see Definition 2below . Theorem 3 also holds for quasi – regular sets , see Definition 2

Before defining coherence, certain problems with sets of measure zeromust be dealt with.For $\in R$, $and\ \ \delta>0$, let $I_\delta(x)$be the open interval $(x-\delta,x+\delta)$ and let $I_\delta=I_\delta(0)$. Suppose that$E_r\subset R^1$is$\mu_{1-\varepsilon}$-measurable , with $0<\mu_{1-\varepsilon}(E_r)<\infty$. The upper density of E_r at x is defined by

$$\sum_{r\geq 1}\overline{D^{1-\varepsilon}}(E_r,x)=\limsup_{j\to 0}\sum_{r\geq 1}\frac{\mu_{1-\varepsilon}(\ E_r\cap I_j\ (x))}{(2j)^{1-\varepsilon}} \tag{1.3}$$

Then $\sum_{r\geq 1}\overline{D^{1-\varepsilon}}(E_r,x)=0$ for $\mu_{1-\varepsilon}-a.e\,.x\notin\ E_r$. And for $\mu_{1-\varepsilon}-a.e\,.1-\varepsilon\in E_r$ one has that$2^{1-\varepsilon}\leq\overline{D^{1-\varepsilon}}(E_r,x)\leq 1$. So $E_r$agrees$\mu_{1-\varepsilon}-a.e.$with its " Lébesgue set "

$$\sum_{r\geq 1}E_r^*=\sum_{r\geq 1}\{x\in E_r\colon 2^{1-\varepsilon}\ \leq\overline{D^{1-\varepsilon}}(E_r,x)\leq 1\}.$$

It is not really necessary that E_r have finite measure . Given$x\in R$, let $(E_r)_x=E_r\cap(-\infty,x]$. It will always be assumed that$\mu_{1-\varepsilon}((E_r)_x)<\infty$ for some x , for otherwise$g.h.i$ is trivial . Let $s=\sup\sum_{r\geq 1}\{x\colon\mu_{1-\varepsilon}(E_r)_x<\infty\}$. Notice$(E_r)_s$ is σ^r – finte with respect to $\mu_{1-\varepsilon}$, so the result above still apply , $(E_r)_s$ agrees $\mu_{1-\varepsilon}-a.e$ with $(E_r)_s$. Let$\sum_{r\geq 1}E_r{}^0=\sum_{r\geq 1}(E_r)_s$ and $\sum_{r\geq 1}(E_r^0)_x=\sum_{r\geq 1}E_r^0\cap(-\infty,x)$. Given sets $A$and A + ε , let $2A+\varepsilon=\{2a+\varepsilon\colon a\in A\}$.

Definition 1 : Let $E_r\subset R$is coherent if there is a constant C^rsuch that for all$x\leq\delta$

$$\limsup_{\delta\to 0}\sum_{r\geq 1}|(E_r^0)_x+I_\delta|\,\delta^{\varepsilon}\leq\sum_{r\geq 1}C^r\mu_{1-\varepsilon}(E_r^0)_x \tag{1.4}$$

This definition depends on the value of $1-\varepsilon$, which will normally be understood. If there is any risk of confusion we will call the set $(1-\varepsilon)$ -coherent . The inequality in the definition can always be reversed (if$\sum_{r\geq 1}C^r=1$) by the definition of Hausdorff measure . The right – hand side is equal to $\sum_{r\geq 1}C^r\,.\mu_{1-\varepsilon}(E_r)_x$ and to$\sum_{r\geq 1}C^r-\sigma_{1-\varepsilon}^r(E_r,x)$. It is necessary to use $\sum_{r\geq 1}(E_r^0)_x$ rather than$\sum_{r\geq 1}(E_r)_x$ because sets of measure zero could greatly affect the left – hand side.

The results in this paper (see [11]) appear with the Fourier transform on the right - hand side, though it is more usual to have it on the left. It makes little difference when$\varepsilon=1$, at least in the periodic case, or when $\varepsilon=0$, but for dimensions in between it matters, because Fourier inversion is not clear. Also, in the case $\varepsilon=1$ it matters for almost –periodic functions. Each of these functions defines a unique Fourier series, but that series does not converge to unique function in the $B^{1+\varepsilon}\ a.p.$ .pseudonorm [3] .

The fundamental case is$\varepsilon=1$. The$M.P.S$ result is the important subcase in which the Fourier transform of the zero – dimensional measure is periodic.The immediate corollary is a proof of the celebrated Littlewood conjecture for trigonometric polynomials. In the same way, an immediate corollary of$g.h.i.$ is an$(\varepsilon+1)$ -dimensional version of Littlewood'sconjecture.

The right –hand side of (1.2) is a natural substitute for the L^1 norm of the Fourier transform of an $(\varepsilon+1)$ - dimensional measure . It resembles terms studied in [3,10] , for example . However, it is usually impossible to compute exactly, and difficult even to determine whether it is finite. For asimple application of $g.h.i.$, let $f^r=\chi_{E_r}$, where E_r is an$(1+\varepsilon)$- coherent or quasi – regular set , for example , a cantor set contained in the unit interval of dimension $\varepsilon+1$. Then (1.2)show that (see [11])

$$\lim_{L\to\infty} \inf L^{\varepsilon} \sum_{r\geq 1} \int_{-L}^{L} \left|\chi_{E_r} \widetilde{d\mu_{\varepsilon-1}}\right| dx = +\infty \tag{1.5}$$

This result is non-trivial, the liminf can converge to 0if E_r is not coherent.
see (4.3). However, it may not be best-possible in the sense thata smaller exponent of$L$on the left-hand side might produce the same result. The question of sharpness seems to be more complicated in this context than in the L^2setting (see [10]).
The results of this paper (see [11]) are organized in the following manner. Section 2is devoted to establishing g.h.i. for the integer dimensions $\varepsilon = 0$ *and* $\varepsilon = 1$ which are Theorems 1 and 2, respectively. It should be noted that lim inf can be replaced by lim in these dimensions. Section 3 discuses $g.h.i$ for themore difficult case $0 < \varepsilon < 1$.
The extensions of Plancherel's theorem by Strichartz [10] involve smoothing out the distribution, applying the classical Plancherel theoremand approximation arguments. Approximation arguments are used inTheorems 1 of this paper to handle the 0-dimensional case. Theproofs for the general cases, Theorems 2 and 3, are from the ground up inthe sense that the $M.P.S.$ machinery is modified for this setting while the$M.P.S.$ result is not used directly.
2-Thefirst Theorem uses the class $B.a.p$of almost periodic function discussed in [2]Besicovitch. These are the almost – periodic functionsu^r for which the pseudonormlim$_{L\to\infty}$ sup $L^{-1}\int_{-L}^{L}\sum_{r\geq 1}|u^r|\,dx$ is finite. If u^r is almost – periodic, then the limit of the right – hand side exists, so lim sup may be replaced by lim . Every trigonometric polynomial is almost –periodic and is in $B.a.p.$ The Fourier series of a $B.a.p.$ functionsu^rconverges to u^r in the pseudonorm above , but may also converge to other $B.a.p$functions – the series does not determine u^r.

Theorem 1: Let$\sum_{r\geq 1} f^r d\mu_0$be a zero – dimensional measure defined by

$$\sum_{r\geq 1} f^r(x) = \sum_{r\geq 1}\sum_{\varepsilon=0}^{\infty} C^r_{1-\varepsilon}\delta(x - a_{1-\varepsilon})$$

where$a_1 < a_2 \ldots <$ and δ is the usual Dirac measure at zero . Assume $\sum_{r\geq 1}\hat{f}^r\, d\mu_0 = \sum_{r\geq 1} c^r{}_{1-\varepsilon} e^{ia_{1-\varepsilon}x}$ belongs to $B.a.p$. Then

$$\sum_{r\geq 1}\sum_{\varepsilon=0}^{\infty}\left|\frac{C^r_{1-\varepsilon}}{1-\varepsilon}\right| \leq \lim_{L\to\infty} L^{-1}\sum_{r\geq 1} C^r \int_{-L}^{L} |f_r \widehat{d\mu_0}(x)|dx \tag{2.1}$$

Proof : First assume that $[a_{1-\varepsilon}]$is a finite sequence with N terms that $\hat{f}^r d\mu_0$ is a polynomial . Let$\varepsilon > 0$. By a lemma of Dirichlet , there are infinitely many integersL_i with numbers $\{a'_{1-\varepsilon}\} \in Z/L_i$ such that $|a_{1-\varepsilon} - \acute{a}_{1-\varepsilon}| < \varepsilon/L_i$ for all $1 - N \leq \varepsilon \leq 1$.
Let $\sum_{r\geq 1} u^r_i(x) = \sum_{r\geq 1}\sum C^r_{1-\varepsilon} e^{id_{1-\varepsilon}x}$. Then for $x \in [-L_i, L_i] u^r_i(x)$

$$\sum_{r\geq 1}\left|\hat{f}^r d\mu_0 - u^r_i(x)\right| \leq \sum_{r\geq 1} C^r \sum |c^r_{1-\varepsilon}||a_{1-\varepsilon} - \acute{a}_{1-\varepsilon}|\,|x| \leq \sum_{r\geq 1} C^r_\varepsilon \sum |c^r_{1-\varepsilon}|$$

Sinceu^r_i is periodic we may apply $M.P.S$

$$\sum_{\varepsilon=0}\sum_{r\geq 1}\left|\frac{c^r_{1-\varepsilon}}{1-\varepsilon}\right| \leq \sum_{r\geq 1} C^r L_i^{-2}\int_{-L_i}^{L_i}|u^r_i(x)|dx \;\; \leq \sum_{r\geq 1} C^r L_i^{-1}\int_{-L_i}^{L_i}|\hat{f}^r d\mu_0|dx \sum_{r\geq 1} C^r_\varepsilon \sum |c^r_\varepsilon|$$

Taking limits as$i \to \infty$ and then as $\varepsilon \to 0$ proves (2.1) in this case .
For the general case , we will approximate using Bohner – Fejer polynomials. [3] Given $\sum_{r\geq 1} u^r = \sum_{r\geq 1}\hat{f}^r d\mu_0(x) \in B.a.p.$, there exists a sequence of polynomials$\{\sigma^r_n\}$ of the form

$$\sum_{r\geq 1}\sigma^r_n(x) = \sum_{r\geq 1}\sum_{\varepsilon=0}^{N(n)} c^{r(n)}_{\varepsilon-1} e^{ia_{1-\varepsilon}x}$$

(where the frequencies$\sigma_{1-\varepsilon}^{r}\ a_{k}$ are the same as those of$\hat{f}^{r}\,d\mu_{0}$) such that

$$\sum_{r\geq 1}\|u^{r}\sigma_{n}^{r}\|_{Bap} = \lim_{L\to\infty}\sup L^{-1}\int_{-L}^{L}\sum_{r\geq 1}|u^{r}\sigma_{n}^{r}|\,dx \leq 2^{-n} \tag{2.2}$$

and

$$\sum_{r\geq 1}\lim_{n\to\infty} c_{\varepsilon-1}^{r(n)} = \sum_{r\geq 1} c_{\varepsilon-1}^{r} \qquad for\ each\ \ \varepsilon-1\ \ . \tag{2.3}$$

We have proven the theorem for suchpolynomials. So using Fatou'sLemma for sums, and the fact that limits exist for $B.a.p.$,functions

$$\sum_{r\geq 1}\sum_{\varepsilon=0}^{\infty}\frac{c_{\varepsilon-1}^{r}}{\varepsilon-1}\leq \inf\sum_{r\geq 1}\frac{c_{\varepsilon-1}^{r}}{\varepsilon-1}\quad\sum_{\varepsilon=0}^{N(n)}\frac{c_{\varepsilon-1}^{r}}{\varepsilon-1}\leq \lim_{n\to\infty}\inf\lim_{L\to\infty}L^{-1}\sum_{r\geq 1}C^{r}\int_{-L}^{L}|u^{r}|\,|u^{r}-\sigma_{n}^{r}|$$

This proves the theorem .

Given$x\in R$,$0\leq\varepsilon\leq 1$, and a set $E_{r}\subset R$, let

$$\sum_{r\geq 1}\sigma_{1-\varepsilon}^{r}(E_{r}\,,x)=\sum_{r\geq 1}\mu_{1-\varepsilon}\,(E_{r}n(-\infty,x])$$

where$\mu_{1-\varepsilon}$ is Hausdorff measure . In the last theorem, the index $\varepsilon-1$ could be written as $\sigma_{0}^{r}(\{a_{\varepsilon-1}\},x)$. The next theorem provides a one dimensional analog. This result should be compared for $\varepsilon=0$.

Theorem 2 : There is an absolute constant C^{r} such that if $\breve{u}^{r}\in L^{1}(R)$

$$\int_{R}\sum_{r\geq 1}\frac{|u^{r}(x)|}{\sigma_{1}^{r}(E_{r}\,,x)}\leq\sum_{r\geq 1}c^{r}\|\breve{u}^{r}\|_{L^{1}} \tag{2.4}$$

Where$E_{r}=supp\ u^{r}$.

Proof : We claim that it is enough to prove (2.4) with$\sigma_{1}^{r}(E_{r}\,,x)$ replaced by $\sum_{r\geq}\sigma_{1}^{r}(E_{r}\,,x)$ +1.To prove this claim , assume (2.4) with $\sum_{r\geq 1}\sigma_{1}^{r}(x)+1$,in the denominator . Given $\breve{u}^{r}\in L^{1}$ and $\varepsilon<1$, $\sum_{r\geq 1}\ \ \breve{v}^{r}(x)=\sum_{r\geq 1}\ \ (1-\varepsilon)\breve{u}^{r}\ ((1-\varepsilon)x$. So,
$\sum_{r\geq 1}\ \ v^{r}(x)=\sum_{r\geq 1}\ \ u^{r}(x,y)\ and\ \sum_{r\geq 1}\ \ \sigma_{1}^{r}(\sup\ v^{r},(1-\varepsilon)-(1-\varepsilon)\sigma_{1}^{r}(E_{r}\,,x))$.
By changing variables and applying (2.4) to$\hat{v}$^r , we get.

$$\int_{R}\sum_{r\geq 1}\frac{|u^{r}(x)|dx}{(1-\varepsilon)^{-1}+\sigma_{1}^{r}(E_{r},x)}\leq\int_{R}\sum_{r\geq 1}\frac{|\,v^{r}(x)|dx}{1+\sigma_{1}^{r}(\sup(1+\varepsilon)\,v^{r}\,,x)}\leq\sum_{r\geq 1}\quad C^{r}\|\tilde{v}^{r}\|_{L^{1}}$$
$$=\sum_{r\geq 1}\quad C^{r}\|\tilde{u}^{r}\|_{L^{1}}$$

and let$\varepsilon\to\infty$ to get (2.4) for$\tilde{u}^{r}$ without the$+1$ in the denominator .

Now the idea of the proof is the same as in $M.P.S$.we will construct functions F_{m}^{r} on Rsuch that

(1) $\sum_{r\geq 1}$ $\hat{F}_{m}^{r}$is supported in $(-\infty,\mathrm{N(m)}]$where $N(m)\to\infty\ as\ m\to\infty$.

(2) $\sum_{r\geq 1}$ $|F_{m}^{r}|_{\infty}\leq 1$.

(3) $30\,.Re\sum_{r\geq 1}\ \ \widetilde{F_{m}}^{r}(x)\geq\sum_{r\geq 1}\ \ |u^{r}(x)|/1+\sigma_{1}^{r}(E_{r},x)$for all $x\in E_{r}\cap(-\infty,N(m)]$.

Given suchF_{m}^{r} , the theorem follows easily if $\sum_{r\geq 1}$ supp $u^{r}\subset(-\infty,N(m)]$ for some m then

$$\int_{R}\sum_{r\geq 1}\frac{|\,u^{r}(x)|dx}{1+\sigma_{1}^{r}(E_{r},x)}$$
$$\leq\sum_{r\geq 1}\quad C^{r}\!:Re\int\sum_{r\geq 1}\quad u^{r}(x)F_{m}^{r}\,dx$$
$$=\sum_{r\geq 1}\quad C^{r}\!:Re\int\sum_{r\geq 1}\quad U^{r}*F_{m}^{r}(m)\,dx\leq\sum_{r\geq 1}\quad C^{r}\|\breve{u}^{r}\|_{L^{1}}\|F_{m}^{r}\|_{L^{\infty}}$$
$$\leq\sum_{r\geq 1}\quad C^{r}\|\breve{u}^{r}\|_{L^{1}}$$

An approximation argument shows that this inequality holds for all$\breve{u}^r \in L^1$.
The construction of F_m^r follows $M.P.S$ .But here F_m^r is a function on R instead of a sequence. For the sake of completeness, an outline of the construction follows.
Let $s_0 = \sum_{r\geq 1} \{x \in E_r : 0 \leq \sigma_1^r(x) \leq 1\}$. Let $s_1 = \sum_{r\geq 1} \{x \in E_r : 1 \leq \sigma_1^r(x) \leq 5\}$ .Define $\sum_{r\geq 1} \{S_i^r\}$for $i > 1$ in this manner so that $|S_i^r| = 4^i$and$E_r = \cup S_i^r$ $. Let\ N(m) = \sup \sum_{r\geq 1} S_m^r$.
Define $f_i^r \in L^2(R)$ by

$$\sum_{r\geq 1} \tilde{f}_i^r = 0 \qquad for\ x \notin S_i^r$$

$$\sum_{r\geq 1} \left|\tilde{f}_i^r\right| = 4^{-i} \qquad for\ x \in S_i^r$$

$$\sum_{r\geq 1} \tilde{f}_i^r(x) u^r(x) \geq 0 \qquad for\ all\ x$$

Let

$$\sum_{r\geq 1} h_i^r = \frac{1}{4} \sum_{r\geq 1} (1 + iH^r)|f_i^r| \in L^2$$

Where$H^r$is the Hilbert transform . Notice that $\sum_{r\geq 1} Re\, h_i^r\, |f_i^r|/4$ and $\sum_{r\geq 1} \|h_i^r\|_2 \leq \sum_{r\geq 1} (3/8)\|f_i^r\|_2 = 3.2^{-i-3}$. Also ,supp $h_i^r \subset (-\infty, 0)$. Let$\sum_{r\geq 1} F_{-1}^r = 0$ and for$m \geq 0$, let

$$\sum_{r\geq 1} F_m^r = \sum_{r\geq 1} \left(F_{m-1}^r(x).\exp\left(-h_m^r(x)\right) + \frac{f_m^r}{5} \right)$$

This is a continuous function in $L^2(R)$. It is supported on the union of the supports of the$f_i^r$for $0 \leq i \leq m$, so condition (1) holds . Because$\exp(-x) + x/5 \leq 1$ whenever $0 \leq x \leq 1$ and since $\sum_{r\geq 1} \|f_m^r\|_\infty \leq \sum_{r\geq 1} \left\|\hat{f}_m^r\right\|_1$, induction proves condition (2)

$$\sum_{r\geq 1} |F_m^r(x)| \leq \left(\sum_{r\geq 1} \exp(-|f_m^r(x)|) + \sum_{r\geq 1} \frac{|f_m^r(x)|}{5} \right) \leq 1$$

Claim . For $i \leq m < \infty$ and for all $x \in S_i^r$

$$\sum_{r\geq 1} \left|\hat{F}_m^r(x) - \tilde{f}_i^r(x)/5\right| \leq \frac{1}{10} \sum_{r\geq 1} \left|\tilde{f}_i^r(x)\right| \tag{2.5}$$

This inequality is proved in$M.P.S$ .in a slightly different context .
Now for $x \in S_i^r$, we have $\sum_{r\geq 1} \sigma_1^r(x) > 4i/3$. So (2.5) shows that

$$\left(Re \sum_{r\geq 1} \hat{F}_m^r u^r(x) - \sum_{r\geq 1} \hat{f}_i^r u^r(x)/5 \right) \leq \sum_{r\geq 1} \tilde{f}_i^r u^r(x)/10$$

and so

$$\sum_{r\geq 1} R_\varepsilon \hat{F}_m^r u^r(x) \geq \sum_{r\geq 1} \hat{f}_i^r(x)\Big/10 \geq \sum_{r\geq 1} \frac{|u^r(x)|}{30(1 + \sigma_1^r(x))}$$

This holds for all $x \in \cup_{i=0}^m S_i^r$ which is$E_r \cap (-\infty, N(m)]$. This proves condition (3) on F_m^r and complete the proof

(3) . We generalize the previous ones to dimensions between 0 and 1. It requires that the measure is supported on a coherent set(see[11]).

Theorem 3: Suppose $0 < \varepsilon < 1$, $f^r \in L^1(d\mu_{1-\varepsilon})$ is supported on E_r, and E_r is $(1-\varepsilon)$-coherent. Then there is a constant C^r independent off^rsuch that

$$\int \sum_{r\geq 1} \frac{|f^r| d\mu_{1-\varepsilon}(x)}{\sigma_{1-\varepsilon}^r(E_r, x)} \leq \liminf_{L\to\infty} L^\varepsilon \sum_{r\geq 1} C^r \int_{-L}^{L} |f^r \widetilde{d\mu_{1-\varepsilon}}|\, dx \tag{3.1}$$

Proof : The idea of the proof is to construct an auxiliary function F_m^r as in Theorem 2. This seems impossible to do on the given fractalset E_r.So instead, the given measure is approximated using convolutionwith a Schwartz function φ_L^r. Then a sequence $\hat{F}_m^r$is constructed for the newsmoothed-out measure, $\sum_{r\geq 1} \phi_L{}^r * f^r d\mu_{1-\varepsilon}$, on a dense dilation of the integers. Afterthis modified $M.P.S.$

construction, we take the $lim\ inf$ as $L \to \infty$. Most ofthe work occurs at this stage, in Lemmas 4 and 5.

Let ϕ^rbe an even Schwartz function such that

$$\int \sum_{r\geq 1} \phi^r\ dx = 1$$

and

$$\operatorname{supp} \sum_{r\geq 1} \hat{\varphi}^r \subseteq [-1,1]$$

We can not arrange that ϕ^r have compact support, but the following lemma is a substitute [11].

Lemma 1: There are constantsC^r andK_0 such that for all x and for all $\varepsilon > 1$

$$\sum_{|n+x|>\varepsilon-1} \sum_{r\geq 1} |\phi^r(n+x)| < \sum_{r\geq 1} C^r/\varepsilon - 1$$

Proof : Since the sum is over a set of integers , we may assume $0 < x < 1$. Also the condition $|1 + x| > \varepsilon - 1$. Since ϕ^r is a Schwartz function $\phi^r(x) = O(x^2)$. So there is C^r and $\varepsilon - 1$such that

$$\int_{\varepsilon-1}^{\infty} \sum_{r\geq 1} |\phi^r(x)|\, dx < \sum_{r\geq 1} C^r/\varepsilon - 1 \quad for\ \ \varepsilon > 0$$

We may assume the same inequality holds for the Schwartz function ϕ^r .

Then for any $x \in [0,1]$.

$$\int_1^{n+1} \sum_{r\geq 1} |\phi^r(t) - \phi^r(n+x)|\, dt \leq sup_{n<t<n+1} \sum_{r\geq 1} |\phi^r(t) - \phi^r(n+x)|$$

$$\leq \int_n^{n+1} \sum_{r\geq 1} \left|\phi^r(\acute{t})n\right| dt\,.$$

So , by the triangle inequality

$$\sum_{n+x>\varepsilon-1} \sum_{r\geq 1} |\phi^r(n+x)| \leq \sum_{n>\varepsilon-1} \int_n^{n+1} \sum_{r\geq 1} \left(|\phi^r(t)| + |\acute{\phi}^r|\right) dt \leq 2 \sum_{r\geq 1} C^r/\varepsilon - 1$$

$$E_r \cup C_\varepsilon^r$$

Which proves the lemma .

Since E_r is coherent, it is bounded below. Let $m = inf \sum_{r\geq 1} E_r$. Fix$E_r > 0$. It is easy to construct a cantor set with $\mu_{1-\varepsilon}$measure 1 . Such a set is coherent by Theorem 4 . So by dilation and translation , there is an $(1-\varepsilon)$- coherent set $\sum_{r\geq 1} C_\varepsilon^r \subset [m-2, m-1]$ such that $\sum_{r\geq 1} \mu_{1-\varepsilon}(C_\varepsilon^r) = \varepsilon$. Notice that the constant (1.4) is not affected by dilation of the set . Let $E_r^1 = E_r \cup C_\varepsilon^r$. This is also coherent , with a (1.4) constant independent of ε . It will replace E_r until the very last step of the proof of Theorem 3 , in which $\varepsilon \to 0$. We will use the new notation

$$\sum_{r\geq 1} E_{r_x} = \sum_{r\geq 1} E_r^1 \cap (-\infty, x)\ and \sum_{r\geq 1} \sigma_{1-\varepsilon}^r = \sum_{r\geq 1} \mu_{1-\varepsilon}\,(E_{r_x})$$

Suppose a real number M has been chosen such that$\varepsilon \leq \sum_{r\geq 1} \sigma_{1-\varepsilon}^r(M) < \infty$. The next lemma provides a kind of uniformity in the limit in (4.1) that will be useful later [11].

Lemma 2 : There is a $\delta_0 > 0$ such that for all $x \leq M$ and all $0 < \delta < \delta_0$,

$$\sum_{r\geq 1} |(E_r)_x + I_\delta|\delta^\varepsilon < \sum_{r\geq 1} C_{E_r}^r\,(\sigma_{1-\varepsilon}^r(x) + \varepsilon)$$

And

$$\sum_{r\geq 1} |(E_r)_x + I_\delta|\delta^\varepsilon > \sum_{r\geq 1} 1/2\,\left(\sigma_{1-\varepsilon}^r(x) - \varepsilon/2\right)$$

Where $C_{E_r}^r$ is the same constant as in the definition of coherent , it depends only on E_r and $1-\varepsilon$.

Proof : Since$\sum_{r\geq 1}\mu_{1-\varepsilon}(\acute{E}_r\cap(-\infty,M)<\infty$, there is a finite increasing sequence of points$\{x_i\}$ in $(E_r)_M$ such that

$\sum_{r\geq 1}\mu_{1-\varepsilon}(E_r^1\cap(x_i,x_{i+1}))<\varepsilon/4$for all i, including the case $i=0$ for which we adopt the convention $x_0=\infty$ .We set the last term $x_N=M$.

For each x_i, $1\leq i\leq N$ there is by the definition of coherent (1.4), a $\delta_i<\varepsilon$such that whenever$\delta<\delta_i$. Let $\sum_{r\geq 1}\quad Q(\delta,(E_r)_x)$ be the minimal number of intervals of length exactly δ required to cover $(E_r)_x$. Then from the definition of Hausdorff measure

$$\sum_{r\geq 1}\mu_{1-\varepsilon}(E_r)_{x_i}\leq\sum_{r\geq 1}\quad\lim_{\delta\to 0}\delta^{1-\varepsilon}\,Q(\delta,(E_r)_x)\leq 2\sum_{r\geq 1}\lim_{\delta\to 0}\delta^{1-\varepsilon}\,|(E_r)_x+I_\delta|$$

So ,δ_i can be chosen small enough that$\delta<\delta_i$ implies .

$$\sum_{r\geq 1}\mu_{1-\varepsilon}(E_r)_{x_i}\leq 2\sum_{r\geq 1}\quad\delta^{1-\varepsilon}\,|(E_r)_x+I_\delta|+\varepsilon/4$$

Let $\delta_0<\varepsilon$ be the smallest of the δ_i . Suppose $\delta<\delta_i$, and $x\leq M$. Then for some $i\geq 0$, we have $x_i<x\leq x_{i+1}$. So

$$\sum_{r\geq 1}\quad|(E_r)_x\ +I_\delta|\delta^\varepsilon\leq\sum_{r\geq 1}\quad\left|(E_r)_{x_{i+1}}+I_\delta\right|\delta^\varepsilon\leq\sum_{r\geq 1}\quad C_E^r[\,\sigma_{1-\varepsilon}^r(x_{i+1})+\varepsilon/4$$

$$\leq\sum_{r\geq 1}\quad(C_{E_r}^r\sigma_{1-\varepsilon}^r(x)+\varepsilon/4)+\sum_{r\geq 1}C_{E_r}^r\mu_{1-\varepsilon}(E_r^1\cap[x,x_{i+1}])\leq\sum_{r\geq 1}C_{E_r}^r[\sigma_{1-\varepsilon}^r(x)+\varepsilon]$$

Like wise

$$\sum_{r\geq 1}|(E_r)_x-I_\delta|\,\delta^\varepsilon\geq\sum_{r\geq 1}\left|(E_r)_{x_i}\ -I_\delta\right|\delta^\varepsilon>1/2\sum_{r\geq 1}\ [\sigma_{1-\varepsilon}^r(x_i)-\varepsilon/4]$$

$$\geq 1/2\sum_{r\geq 1}\ [\sigma_{1-\varepsilon}^r(x)-\varepsilon/2]$$

Which proves the lemma .

Now fix δ such that $0<\delta<\delta_0$. Fix $K>K_0$ as defined by Lemmas1 . We also assume $\sum_{r\geq 1}\quad KC_4^r>\sum_{r\geq 1}\quad 2C_5^r$, where C_4^r andC_5^r are absolute constants that arise in Lemma 4and 5, respectively . Let $L=K/\delta$ and let $\varphi_L^r(x)=\varphi^r(L_x)$. Let

$$\sum_{r\geq 1}S^r=\sum_{r\geq 1}\quad[(E_r+I_\delta)\cap Z\acute{/}L]\cap(-\infty,M]$$

Lemma 3(see[11]) :There is a sequence$\hat{F}_m^r:Z/L\to\mathbb{C}$ such that

$$Re\sum_{r\geq 1}\quad\left[\hat{F}_m^r(n/L)\,\varphi_L^r*f^r\,d\mu_{1-\varepsilon}(n/L)\right]$$

$$\geq\sum_{r\geq 1}\quad C^rK^\varepsilon\frac{\varphi_L^r*d\mu_{1-\varepsilon}(n/L)}{L^{1-\varepsilon}\left(\varepsilon+\sigma_{1-\varepsilon}^r(n/L)\right)}\tag{3.2}$$

for all $n/L\in S_r$

$$\sum_{r\geq 1}\hat{F}_m^r(n/L)\ \leq\sum_{r\geq 1}\quad\frac{C^rK^\varepsilon}{L^{1-\varepsilon}\left(\sigma_{1-\varepsilon}^r(n/L)\right)+\varepsilon},$$

for all $n/L>m-1/2$ (3.3)

$$\sum_{r\geq 1}\hat{F}_m^r(n/L)\ \leq\sum_{r\geq 1}\quad C_\varepsilon^r\,,\ \ for\ all\ n\tag{3.4}$$

$$\sum_{r\geq 1}\quad\|F_m^r\|_\infty\leq 1\tag{3.5}$$

Proof : Let $\sum_{r\geq 1}\quad S^r=\{n_1/L,n_2/L,\dots,n_{1-\varepsilon}/L\}$. Choose $i_0$such that

$$4^{-i_0}<\varepsilon\leq 4^{-i_{0+1}}\tag{3.6}$$

We can ensure that S^r has at least 4^{i_0} terms by choosing $\delta_0$small enough (to see this consider Lemma 2 and inequality (3.10) below) In fact , we can assume that the first 4^{i_0} terms come from C_ε^r , and are all less than$m$. Let S_0^r be the set of the first 4^{i_0} terms of S^r. Let S_1^r be the set of the next 4^{i_0+1} terms etc , until S^r is exhausted . If there are terms left over when this construction stops , they are included in the last set ,S_t^r.So,S^r =$\cup_{i=0}^{t} S_i^r$ where each $\sum_{r\geq1} S_i^r$,$i < t$, has 4^{i_0+j} elements . Then , construct functions f_i^r and F_m^r as in $M.P.S.$(using the function $\sum_{r\geq1} \varphi_L^r * f^r d\mu_{1-\varepsilon}$ instead of the function referred to there as$\tilde{\varphi}^r$) , so that the following inequality holds

$$\sum_{r\geq1} \left|\hat{F}_m^r(n/L) - \sum_{r\geq1} \frac{\tilde{f}_1^r(n/L)}{5}\right| \leq (1/10)4^{-(j_0+j)} \quad for \;\; n/L\, S_i^r \tag{3.7}$$

The calculations in$M.P.S.$ actually prove something a little more general. If$n < n_{1-\varepsilon}$ and $n/L \notin S^r$ define $i = i(n)$ by

$$n_i/L = \sum_{r\geq1} \min\{n_{1-\varepsilon}/L \in S^r : n_{1-\varepsilon} > n\}$$

Define$i = i(n)$ by the condition $n_i/L \in S_i^r$. Then inequality (3.7) also holds for this $n$and i. However in this case $\tilde{f}_i^r(n/L) = 0$, so

$$\sum_{r\geq1} \hat{F}_m^r(n/L) \leq (1/10)4^{-(i_0+i)} \qquad for\, n/L \in S^r, n < n_{1-\varepsilon}, i > i(n) \tag{3.8}$$

from the construction of S^r above , if$n_{1-\varepsilon}/L \in n\, S_i^r$ with $i > 0$ then

$$\sum_{r\geq1} c_1^r 4^{-(i_0+i)} \leq \frac{1}{\varepsilon - 1} \leq \sum_{r\geq1} c_2^r 4^{-(i_0+i)} \tag{3.9}$$

and since$\sum_{r\geq1} E_{n(1-\varepsilon)/L}^r + I_\delta$, is made up of intervals of length at least $\delta > 1/L$

$$\sum_{r\geq1} c_3^r \frac{(1-\varepsilon)}{L} \leq \sum_{r\geq1} E_{n(1-\varepsilon)/L}^r + I_\delta \leq \sum_{r\geq1} c_4^r \frac{(1-\varepsilon)}{L} \tag{3.10}$$

where the c_i^r in (3.9) and (3.10) are absolute constants from $M.P.S.$

$$\sum_{r\geq1} \left|\tilde{f}_i^r(n/L)\right| = 4^{-(i_0+i)} \quad for \;\; n/L \;\in S_i^r \tag{3.11}$$

For$n/L > m - 1/2$,$\sum_{r\geq1} \sigma_{1-\varepsilon}^r(n/L) \geq \varepsilon$, so that the $-\varepsilon/2$ in Lemma 2 may be replaced by $(1+\varepsilon)$. Also , $i > 0$ for these$n$ so (3.9) applies . From these, and (3.10) , we get (3.3) for $n/L \in S^r$,

$$\sum_{r\geq1} \hat{F}_m^r(n/L) \leq 4^{-(i_0+i)} \leq \frac{1}{\varepsilon - 1} \leq \sum_{r\geq1} \frac{1}{L\left|(E_r)_{n/L} + I_\delta\right|}$$

$$\leq \sum_{r\geq1} \frac{1}{(\varepsilon - 1)L^{1-\varepsilon}\left(\sigma_{1-\varepsilon}^r(n/L)\right) + \varepsilon} \tag{3.12}$$

where we have omitted absolute constants.

This inequality also applies off S^r as follows (see [11]), (3.3) is trivial for$n > n_{1-\varepsilon}$ because $\hat{F}_m^r$ is zero there . For $n/L \notin S^r$and $< n_{1-\varepsilon}$, the first two inequalities of (3.12) holds with$i = i(n)$ and $(\varepsilon - 1) = i(n)$. The third then holds with the subscript$n_{\varepsilon-1}/L$. This change is harmless because

$$\sum_{r\geq1} \left|(E_r)_{n_{1-\varepsilon}} + I_\delta\right| - \sum_{r\geq1}\left|(E_r)_{n/L} + I_\delta\right| \leq \delta$$

So the error in the denominator is at most $L\delta = K$ which is much smaller than $L_\delta^{1-\varepsilon}$, we can assume L is quite large through proper choice of δ_0.So the error is negligible and we have (3.3) for all $n/L > m - 1/2$.

Inequality (3.4) follows from (3.11) and (3.7) on S^r , and from (3.8) off S^r . Part of the$M.P.S.$construction is that $\sum_{r\geq1} \tilde{f}^r(\varphi_L^r * f^r d\mu_{1-\varepsilon}) \geq 0$. This together with (3.11) (3.7) and (3.9) imply that .

$$Re\left[\sum_{r\geq1} \hat{F}_m^r(n/L)\, \varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)\right] \geq \sum_{r\geq1} \frac{|\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)|}{c_5^r K} \quad \text{for } n/L) \in S^r \tag{3.13}$$

except that for $(n/L) \in S_0^r$ we must replace $c_5^r K$ by 4^{i_0} . With (3.10) and Lemma 2 , this shows that the left side of (3.2) is at least

$$\sum_{r\geq 1}\frac{|\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)|}{L|(E_r)_{n/L}+I_\delta|}\geq\sum_{r\geq 1}\frac{|\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)|}{K^\varepsilon L^{1-\varepsilon}\left(\sigma_{1-\varepsilon}^r(n/L)\right)+\varepsilon}$$

for $n/L\in\sum_{r\geq 1}\ (S^r-S_{i_0}^r)$. For $n/L\in S_{i_0}^r$, use (3.6) instead . In this case we need the inequality $>1/K^\varepsilon L^{1-\varepsilon}\varepsilon$, which holds for large enough L . This proves (3.2) for all $n/L\in S^r$.

Inequality (3.5) is part of the $M.P.S.$construction. This proves Lemma 3.

$\hat{F}_m^r$ must be slightly modified off S^r before proceeding with the proof of the theorem . For $n/L>M$ with $n/L\notin S^r$, let

$$\sum_{r\geq 1}\quad G^r(n/L)\,\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)=\frac{|\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)|}{L^{1-\varepsilon}\left(\varepsilon+\sigma_{1-\varepsilon}^r(n/L)\right)}\tag{3.14}$$

Let

$$A(L)=\sum_{r\geq 1}\ \sum_{n\in Z}G^r(n/L)\,\varphi_L^r\ * f^r d\mu_{1-\varepsilon}(n/L)$$

$$(A+\varepsilon)(L)=\sum_{r\geq 1}\ \sum_{n\in Z}\left(\hat{F}_m^r\ -G^r\right)(n/L)\,\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)$$

Since G^r may be viewed as a substitute for $\hat{F}_m^r$, the term $(A+\varepsilon)(L)$ may be viewed as an error term $L^{1-\varepsilon}|A|$ is supposed to approximate the left – hand side of (3.1) or large enough L . This will be the content of lemma 4 . The next calculation shows the relation to the right – hand of (3.1) (see [11]).

$$L^{1-\varepsilon}|A|-L^{1-\varepsilon}|A+\varepsilon|$$

$$\leq L^{1-\varepsilon}\sum_{r\geq 1}\ \sum_{n\in Z}\hat{F}_m^r\ (n/L)\,\varphi_L^r$$

$$* f^r d\mu_{1-\varepsilon}(n/L)=L^{1-\varepsilon}\sum_{r\geq 1}\quad\left|\hat{F}_m^r(\varphi_L^r * f^r\widehat{d\mu_{1-\varepsilon}}(n/L))(o)\right|$$

$$\leq L^{1-\varepsilon}\sum_{r\geq 1}\quad\int\left|\hat{\varphi}^r(x/L)f^r\widehat{d\mu_{1-\varepsilon}}(x)\right|\leq L^{1-\varepsilon}\sum_{r\geq 1}\quad\|\hat{\varphi}^r\|_\infty\int_{-L}^{L}\left|f^r\widehat{d\mu_{1-\varepsilon}}(x)\right|dx\,.$$

Lemma 4:[11] There is an absolute constant C^r such that

$$\lim_{L\to\infty}L^{1-\varepsilon}|A|\geq\sum_{r\geq 1}\quad C_4^r\int_{-\infty}^{M}\frac{|f^r|d\mu_{1-\varepsilon}}{\varepsilon+\sigma_{1-\varepsilon}^r(x)}$$

Lemma 5 :[11] There is an absolute constant C_5^r such that

$$\lim_{L\to\infty}sup L^{1-\varepsilon}|A+\varepsilon|\leq\sum_{r\geq 1}\frac{C_5^r}{K}\int_{-\infty}^{M}\frac{|f^r|d\mu_{1-\varepsilon}}{\varepsilon+\sigma_{1-\varepsilon}^r(x)}$$

Since K was chosen so that $\sum_{r\geq 1}\frac{C_5^r}{K}<\sum_{r\geq 1}\frac{C_4^r}{2}$, we can combine the lemmas to get

$$\sum_{r\geq 1}\quad\left(C_4^r-\frac{C_4^r}{2}\right)\int_{-\infty}^{M}\frac{|f^r|d\mu_{1-\varepsilon}}{\varepsilon+\sigma_{1-\varepsilon}^r(x)}\leq\lim_{L\to\infty}\inf(L^{1-\varepsilon}|A|-L^{1-\varepsilon}\,|A+\varepsilon|)$$

$$\leq\sum_{r\geq 1}\quad\|\hat{\varphi}^r\|_\infty\lim_{L\to\infty}\inf L^\varepsilon\int_{-L}^{L}\left|f^r\widehat{d\mu_{1-\varepsilon}}\right|dx.$$

Notice that $\sum_{r\geq 1}\quad\sigma_{1-\varepsilon}^r(x)=\sum_{r\geq 1}\quad\sigma_{1-\varepsilon}^r(E_r,x)+\varepsilon$ for all x in the support of f^r . Then let $\varepsilon\to 0$ and let $M\to\sup\{x:\sigma_{1-\varepsilon}^r(x)<\infty\}$ to get (3.1) .

Proof of lemma 4: let ε_1 be arbitrary $0<\varepsilon_1<\varepsilon$. Let

$$I_i=\{x\leq M: i\varepsilon_1\leq\sigma_{1-\varepsilon}^r<(i+1)\varepsilon_1\}\tag{3.15}$$

for $i=0,1,\ldots,J$ where $M\in I_i$. Notice that $\sum_{r\geq 1}\mu_{1-\varepsilon}(\acute{E}_r\cap I_i)=\varepsilon_1$ for each $0\leq i<J$. Then by (3.2) and (3.4),

$$L^{1-\varepsilon}|A| = L^{1-\varepsilon}\left|\sum_{r\geq1}\sum_{n\in Z} G^r(n/L)\,\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)\right| \geq \sum_{r\geq1} G^r \sum_{n<LM} \frac{|\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)|}{\varepsilon + \sigma_{1-\varepsilon}^r(n/L)}$$

$$\geq \sum_{r\geq1} G^r \sum_{J=0}^{J} \sum_{n/L\in I_i} \frac{|\varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)|}{\varepsilon + (J+1)\varepsilon_1} = \sum_{r\geq1} G^r \sum_{J=0}^{J} \frac{A_i}{\varepsilon + (J+1)\varepsilon_1}$$

the last equation being a definition of A_i . We now claim that , for all J ,

$$\lim_{L\to\infty} \inf A_i \geq \sum_{r\geq1} \left|\int_{I_i} f^r d\mu_{1-\varepsilon}\right| \tag{3.16}$$

To proof (3.16) notice that for all $J.L$,

$$A_i \geq \sum_{r\geq1} \left|\sum_{n/L\in I_i} \varphi_L^r * f^r d\mu_{1-\varepsilon}(n/L)\right|$$

$$\geq -\sum_{r\geq1} \left|\sum_{n/L\in I_i} \varphi_L^r * \chi_{I_i} f^r d\mu_{1-\varepsilon}(n/L)\right| + \left|\sum_{r\geq1}\sum_{n\in Z} \varphi_L^r * \chi_{I_i} f^r d\mu_{1-\varepsilon}(n/L)\right|$$

$$-\left|\sum_{r\geq1}\sum_{n/L\notin I_i} \varphi_L^r * \chi_{I_i} f^r d\mu_{1-\varepsilon}(n/L)\right| = -A_i^a + A_i^{a+\varepsilon} - A_i^{a+2\varepsilon}$$

Now since $\hat{\varphi}^r \subseteq [-1.1]$. The Poisson summation formula shows that for all x

$$\sum_{z}\sum_{r\geq1} \varphi^r(n-x) = \sum_{r\geq1}{}' \varphi^r(0) = 1$$

So

$$A_i^{a+\varepsilon} = \left|\sum_{z}\sum_{r\geq1} \varphi^r(n/L-x)\, f^r(x) d\mu_{1-\varepsilon}\right| = \sum_{r\geq1} \int_{I_i} |f^r d\mu_{1-\varepsilon}|$$

So , we must show that A_i^a and $A_i^{a+2\varepsilon}$ approach zero as $L\to\infty$. We will assume $I_i = (-\infty, 0)$, other case being similar .

Since φ^r is a Schwartz function , there is a constant R such that for all real$x$

$$\sum_{n\in Z}\sum_{r\geq1} |\varphi^r(n-x)| < R$$

Let $\varepsilon_2 > 0$. Since $f^r \in L^1(d\mu_{1-\varepsilon})$and $\varepsilon < 1$, there is a $\acute{\delta} > 0$ such that

$$\int_{I_\delta(0)} \sum_{r\geq1} f^r d\mu_{1-\varepsilon} < \varepsilon_2/R \tag{3.17}$$

of course , ifI_i has a boundary point at some $x_0 \neq 0$, then$I_\delta(0)$ must be replaced by $I_\delta(x_0)$. Sinceφ^r is a Schwartz function , there is a constant C^rsuch that

$$\sum_{r\geq1} \varphi^r(L(x-n/L) \leq \sum_{r\geq1} \frac{C^r}{|L(x-n)/L|^2+1}$$

Since $f^r \in L^1(d\mu_{1-\varepsilon})$

$$\sum_{r\geq1}\sum_{n\leq0} \left|\varphi_L^r * (\chi_{[\acute{\delta},\infty)} f^r d\mu_{1-\varepsilon})(n/L)\right| \leq \sum_{r\geq1}\sum_{n\leq0} \frac{C^r}{\left|L(\acute{\delta}-n)/L\right|^2+1} \leq \sum_{r\geq1} C^r(L\acute{\delta})^{-1}$$

which approaches zero as L approaches infinity . Also

$$\sum_{r\geq 1}\sum_{n\leq 0}\left|\varphi_L^r*(\chi_{[0,\acute{\delta}]}f^r\,d\mu_{1-\varepsilon})(n/L)\right|\leq\sum_{r\geq 1}\left(\int_0^{\acute{\delta}}\left(\sum_{n\leq 0}|\varphi_L^r(n/L-x)|\right)f^r\,d\mu_{1-\varepsilon}(x)\right)$$
$$\leq R(\varepsilon_2/R)=\varepsilon_2$$

Which shows that

$$\limsup A_i^a\leq \lim \sum_{r\geq 1}\sum_{n\leq 0}\left|\varphi_L^r*(\chi_{[0,\infty)}f^r\,d\mu_{1-\varepsilon}(n/L)\right|\leq 0+\varepsilon_2$$

as $\rightarrow\infty$. Since$\varepsilon_2>0$ was arbitrary , this proves that$\lim A_i^a=0$. This proof work because $n\in(-\infty,0)$ and $x\in(0,\infty)$ range over disjoint intervals . Soit works for arbitrary intervals I_i and for $A_i^{a+2\varepsilon}$ as well . The claim is proved , but it is not exactly what we need the absolute value should be inside the integral . We now show that the error is small. Define

$$e_i=\sum_{r\geq 1}\left|\int_{I_i}f^r\,d\mu_{1-\varepsilon}\right|\geq 0$$

We will show that $\sum e_i\rightarrow 0$ as $\varepsilon_1\rightarrow 0$. This will complete the proof of Lemma 4.

$$\liminf L^{1-\varepsilon}|A|\geq\liminf\sum_{r\geq 1}C^r\sum\frac{A_i}{\varepsilon+(J+I)}\geq\sum\frac{A_i}{\varepsilon+J\varepsilon_1}$$
$$\geq\sum_{\liminf r\geq 1}C^r\sum\frac{A_i}{\varepsilon+J\varepsilon_1}\quad\geq\sum_{r\geq 1}C^r\frac{\int_{I_i}f^r\,d\mu_{1-\varepsilon}-e_i}{\varepsilon+J\varepsilon_1}$$
$$\geq\sum_{r\geq 1}C^r\frac{\int_{-\infty}^{M}|f^r|\,d\mu_{1-\varepsilon}}{\varepsilon+\sigma_{1-\varepsilon}^r(x)}-\sum\frac{e_i}{\varepsilon}\tag{3.18}$$

Let $\mu=\sum_{r\geq 1}\chi_{E_{r_M}}\mu_{1-\varepsilon}$ and for each interval $I\subset R$ define

$$\sum_{r\geq 1}avg_1(f^r)=\frac{I}{\mu(I)}\int_I\sum_{r\geq 1}f^r\,d\mu$$

and

$$U\sum_{r\geq 1}f^r(x)=(\varepsilon_1)^{-1}\sup\int_I\sum_{r\geq 1}|f^r-avg_1(f^r)|\,d\mu\tag{3.19}$$

Taken over all intervals I containing x such that$\mu(I)=\varepsilon_1$. It is easy to check that U is a sublinear operator on $L^1(d\mu)$ and that for $x\in I_i Uf^r(x)\geq e_i/\mu(I_i)$. So

$$\sum e_i=\sum\frac{1}{\mu(I_i)}\int_{I_i}e_i\,d\mu\leq\int_R\sum_{r\geq 1}U\,f^r(x)d\mu\tag{3.20}$$

If f^r is continuous with compact support , then the right – hand side of (3.20) goes to zero withε_1 . For in that case f^r is uniformly continuous and$\sum_{r\geq 1}|f^r(x)-avg_1(x)|\rightarrow 0$ uniformly in$x$ and$I$ as $|I|=\varepsilon_1\rightarrow 0$. Therefore .$Uf^r\rightarrow 0$uniformly on its support , which is bounded . Since μ is finite on any bounded set ,$\int\sum_{r\geq 1}Uf^r\rightarrow 0$ in this case .

Now we show that U is bounded on$L^1(d\mu)$independent of ε_1. Given any $x\in E_r$, there is a$J=J(x)$ such that $x\in I_i$. Let$I_i{}^*=I_{i-1}\cup I_i\cup I_{i+1}$(where$I_{i-1}$ is the emptyset) Appling the triangle inequality to (3.19) , Uf^r splits naturally into two parts , each of which is at most $\sum_{r\geq 1}Vf^r(x)=(\varepsilon_1)^{-1}\int_{I_{i(x)}{}^*}\sum_{r\geq 1}|f^r|d\mu$. But $Vf^r(x)$ is constant on each $I_i$so ,

$$(1/2)\sum_{r\geq1}\int_R U\,f^r(x)d\mu \leq \sum_{r\geq1}\int_R V\,f^r(x)d\mu = \sum_{r\geq1}\ \sum\int_{I_i} V\,f^r(x)d\mu \leq \sum_{r\geq1}\sum\int_{I_i^*}|f^r|\,d\mu$$

$$= 3\sum_{r\geq1}\sum\int_{I_i}|f^r|\,d\mu = 3\sum_{r\geq1}\int_R|f^r|\,d\mu$$

which shows U is bounded on L^1 .

For any $f^r \in L^1(d\mu)$, there is a continuous function g^r with compact support that is arbitrarily close to f^r in the $L^1(d\mu)$. So ,

$$\sum_{r\geq1}\int U\,f^r(x)d\mu \leq \sum_{r\geq1}\int U\,g^r(x) + U(f^r - g^r)(x)d\mu$$

$$\leq \sum_{r\geq1} U\,g^r(x)d\mu + 2\sum_{r\geq1}\int|(f^r - g^r)|(x)\,d\mu \qquad (3.21)$$

The first term of the last expression goes to zero with ε_1 because g^r is continuous . The second term can be made arbitrarily small by proper choice of g^r . With (3.20) , this completes the proof that $\sum e_i \to 0$, and also the proof of Lemma 4.

Proof of Lemma 5: Let $H^r(n/L) = \sum_{r\geq1}(\ \hat{F}_m^r - C^r)(n/L)$, which is zero for $n/L \in S^r$. For $n/L > m - 1/2L^{1-\varepsilon}\,|H^r(n/L)| \leq \sum_{r\geq1}\ C^r((\varepsilon + \sigma_{1-\varepsilon}^r(n/L))^{-1}$by (3.3) and (3.14) . Also assuming$L$ is large enough ,$|H^r(n/L)| < \varepsilon$ for all n.

Let $\{I_i\}$ be the partition defined in Lemma 4 except that now $\varepsilon_1 = \varepsilon$. Similar to proof that $A_i^{1-\varepsilon} \to 0$, we see that for each i

Using Lemma 1,

$$\sum_{r\geq1}\ \sum_{n/L\in I_{i-S^r}}|\varphi_L^r * \chi_{I_i}f^r d\mu| \leq \sum_{r\geq1}\ \sum_{n/L\notin S^r}\left\|\left\|\varphi_L^r * \chi_{I_i}f^r d\mu\right\|\right\|_{L^\infty(E_r)}\left\|\chi_{I_i}f^r\right\|_{L^1(d\mu_{1-\varepsilon})}$$

$$\leq \sum_{r\geq1}\frac{C^r}{K}\left\|\chi_{I_i}f^r\right\|_{L^1(d\mu_{1-\varepsilon})}$$

because$n/L \notin S^r$ and $x \in E_r$ implies $|n/L - x| > \delta$ so that $|n/L - x| > L\delta = K$. So for each J

$$\lim_{L\to\infty} sup\sum_{r\geq1}\ \sum_{n/L\in I_{i-S^r}}|\varphi_L^r * f^r d\mu| \leq \sum_{r\geq1}\frac{C^r}{K}\int_{I_i}|f^r|\,d\mu_{1-\varepsilon}$$

Notice that H^r is zero above $M = sup\ I_i$. So summing over J gives

$$\lim_{L\to\infty}\sup L^{1-\varepsilon}\sum_{r\geq1}\ \sum_{n/L>m-1/2}|(H^r(n/L)|.\,|\varphi_L^r * f^r d\mu_{1-\varepsilon}| \leq \sum_{r\geq1}\frac{C^r}{K}\sum_{i=0}^{J}\frac{\int_{I_i}|f^r|\,d\mu_{1-\varepsilon}}{\varepsilon + \sigma_{1-\varepsilon}^r(inf I_i)}$$

$$\leq \sum_{r\geq1}\frac{C^r}{K}\int_{-\infty}^{M}\frac{|f^r|d\mu_{1-\varepsilon}}{\varepsilon + \sigma_{1-\varepsilon}^r(x)}$$

Because$\varepsilon + \sigma_{1-\varepsilon}^r(x)$ is roughly constant on each I_i(except when $J = 0$, in which case the numerator is zero) .

For $n/L \leq m - 1/2$, we use the fact that$|(H^r(n/L)| \leq \varepsilon$ and that for $x \in suppf^r$, $(x - n/L) > 1/2$. So

$$L^{1-\varepsilon}\sum_{r\geq 1}\sum_{n/L\leq m-1/2}|(H^r(n/L)|.|\varphi_L^r * f^r d\mu_{1-\varepsilon}|$$
$$\leq \sum_{r\geq 1}\varepsilon L^{1-\varepsilon}\left\|\sum_{n/L\leq m-1/2}|\varphi_L^r|(n/L-x)\right\|_{L^\infty(E_r)}\|f^r\|_1$$
$$\leq L^{1-\varepsilon}\left\|\sum_{|n-Lx|\geq L/2}|\varphi^r|(n-Lx)\right\|_{L^\infty(E_r)}\|f^r\|_1$$

which goes to zero as $L\to\infty$ because
$\left\|\sum_{|n-Lx|\geq L/2}|\varphi^r(n-Lx)|\right\|_{L^\infty(E_r)} < \sum_{r\geq 1}\frac{C^r}{L}$by Lemma 1 . This proves Lemma 5

Theorem4:[11]There is an absolutemaximal constantC^r such that

$$\lim_{L\to\infty} sup\, L^{1-\varepsilon}|A+\varepsilon| \leq \sum_{r\geq 1} C^r \int_{-\infty}^{M}\frac{|f^r|d\mu_{1-\varepsilon}}{\varepsilon+\sigma_{1-\varepsilon}^r(x)} \leq \lim_{L\to\infty} inf\, L^{1-\varepsilon}|A|$$

Proof: Appling Theorem 3 and Lemma 4 we show that , for$C^r = \max\{\frac{C_5^r}{K}, C_1^r\}$ with the result given by the approximated inequality say$L^{1-\varepsilon}|A| \leq C^r$
The required inequality is obtained after taking the infinium and supremum over all $L^{1-\varepsilon}$whereC_5^r define Lemma 4, we get from Lemmas 3 and 4

$$\sum_{r\geq 1}(C_4^r - C_4^r/2)\int_{-\infty}^{M}\frac{|f^r|d\mu_{1-\varepsilon}}{\varepsilon+\sigma_{1-\varepsilon}^r(x)} \leq \lim_{L\to\infty} inf\,(L^{1-\varepsilon}|A| - |A+\varepsilon|)$$
$$\leq \sum_{r\geq 1}\|\varphi^r\|_\infty \lim_{L\to\infty} inf \int_{-L}^{L}\left|\hat{f}^r d\mu_{1-\varepsilon}\right| dx$$

Proposition 1:[11]. Given $0<\varepsilon<1$, there is a set $E_r \subset [0,1]$ that is a-coherent but not quasi-regular.

Proof. Given a positive integer k, construct a cantor set, $C^r(2^k, 3^k)$, as follows. Remove2^k-1 intervals of equal length from $[0,l]$ leaving 2^ksubintervals, each of length 3^{-k}. Repeat the excision on each of the 2^ksubintervals leaving 2^k subintervals of length 3^{-2k}. Repeat adinfinitum, so that after stage 1 the setC_1^r, has 2^{kI} subintervals, each of length3^{-kI}. Let $C^r(2^k,3^k) = \cap\, C_1^r$. For every k, this set has dimension $\varepsilon = 1 - ln2/ln3$. Notice that $C^r(2,3)$ is the usual cantor $2/3$ set.

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