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The Accuracy of Euler and modified Euler Technique for First Order Ordinary Differential Equations with initial condition

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Abstract: *The study concerned the accuracy of numerical method in solving first order ordinary differential equations. Euler and Modified Euler techniques have been implemented using different step size h and time t and the local truncation errors have been calculated. It found that the numerical solutions are not close to analytical and unstable when we increased the time t, and the local truncation errors are unreasonable.*

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I. INTRODUCTION

Many models from Engineering and Sciences are used ordinary differential equation in their applications, so the needs for study of ordinary differential equation is essential to understand the behavior of solution methods.

Ordinary differential equations are equation where at unknown is a function in one variable . We can classify the ordinary differential equations according to their properties. Among these characteristics are form , order linearity or type of given boundary values. In other word an equation which involves the differential coefficients of one variable is called ordinary differential equation.

The solution to a differential equation is the function that satisfies the differential equation and also satisfies certain initial conditions on the function. In solving a differential equation analytically, we usually find a general solution containing arbitrary constants and then evaluate the arbitrary constants so that the expression agrees with the initial conditions. Many analytical techniques exist for finding the solution of such equations. Besides all these techniques, sometimes it happens that a problem cannot be solved at all or lead to solutions which are so difficult to obtain. In such case the numerical technique is useful.

In this work, we are going to study the accuracy and behavior of first order linear differential equation using Euler and Euler Modified techniques for different step size and time.

II. DEFINITIONS AND GENERAL CONCEPTS OF DIFFERENTIAL EQUATIONS

2.1.Definition: An equation involving a function $y(t)$ of one independent variable (t) and its derivatives $y'(t) \dots\dots\dots. y^{(n)}(t)$ is called an ordinary differential equation (ODE) of order n .

2.2.Definition: An ordinary differential equation in general form $F\left(t, y, y'(t), y''(t) \dots\dots y^{(n)}(t)\right) = 0$ is called ODE in implicit form. In special form $y^{(n)} = f((t, y, y', y'' \dots\dots\dots, y^{(n-1)})$ is called an ODE in explicit form.

2.3.Definition: An ordinary differential equation is called linear if y and its derivatives appear only as linear terms.

2.4.Definition: An ordinary differential equation of order n for a function $y(t)$ together with n initial conditions (IC) at single point is t_0 called an initial value problem (IVP), $y(t_0) = y_0, y'(t_0) = y_1 \dots\dots y^{(n-1)} = y_{n-1}$.

2.5.Definition: An ordinary differential equation of order n together with n boundary conditions (BC) at least two different point t_0, t_1 is called a boundary value problem (BVP).

2.6.Definition: A set of first order ordinary differential equation:

$y' = \mathrm{f}(\mathrm{t}, \mathrm{y})$

$y_1{}' = f_1(t, y_1 \dots \dots . . y_n)$
$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$
$y'_n = f_n(t, y_1 \dots \dots . . y_n)$
is called a first order system of ordinary differential equations.
2.7.Definition: A set of first order of ordinary differential equations is called autonomous if all functions f_k on the right-hand side do not depend on t
$y' = f(y)$ or
$y_1{}' = f_1(y_1, \dots \dots \dots \dots y_n)$
....
$y_n{}' = f_n(y_1, \dots \dots \dots \dots y_n)$
2.8.Definition: A solution of a system of ordinary differential equation in the form
$y(t) = y_0$ with $y_0 \in \mathbb{R}^2$ is called an equilibrium solution.
$y(t) = y(t + T)$ with $T > 0$ is called a periodic. The parameter T denotes period.
2.9.Definition: An equilibrium solution $y(t) = y_0$ of an autonomous system is called stable if any solution in the neighborhood of y_0 will always approach this equilibrium solution as $t \to \infty$. In this case $y(t) \to y_0$ for $t \to \infty$.
Otherwise the equilibrium solution is called unstable.

III. NUMERICAL CONCEPT OF ORDINARY DIFFERENTIAL EQUATIONS

Many numerical methods generate sequence of points as solution, so numerical solution is calculated only at discrete points.
3.2.Definition: The transformation of a continuous solution $y(t)$ for $t \in [0, \infty]$ into a discrete sequence of points (y_k) for $k \in N_0$ at points in time t_k with
$y_k = y(t_k) \approx w_k$
is called discretization. If the points in time t_k are equidistant with $h > 0$, then $t_k = h_k$. Numerical approximation of y are denoted by w .
The numerical solutions errors has two main sources, firstly any numerical algorithm is an approximate extracted from discretization which cause truncation errors. Second on computer, number are stored in a fixed number of digits. Any floating -points operation thus cause round off errors.
3.3.Definition: The errors of a numerical methods caused by discretization and depend on the step size h and contribute to total errors, is called truncation errors.
The limited machine precision is called round off error.
The approximation quality of numerical solution is a central issue in the context of initial value problem. The most important error that describes the difference between the exact and numerical solution. The local error is defined as difference between numerical and exact solution in one single step and easier to compute.
3.4.Definition:[4]. The local truncation error $E(t, h)$ at time t with step size h is defined as the difference between the numerical solution $w(t, h)$ and the exact solution $y(t)$: $E(t, h) = w(t, h) - y(t)$.
3.5.Theorem: If the function p and g are continuous on an open interval I: $\propto < t < \beta$ containing the point $t = t_0$, then there exist a unique function $y = \varphi(t)$ that satisfies the differential equation

$$y' = p(t)y = g(t) \qquad (3.1)$$

for each t in I and that also satisfies the initial condition

$$y(t_0) = y_0 \qquad (3.2)$$

where y_0 is an arbitrary prescribed initial value.
The theorem asserts both the existence and uniqueness of the solution of the initial value problem.

IV. FINITE DIFFERENCE APPROXIMATION

From the definition of derivative

$$f'(x) = \lim_{n\to 0} \frac{f(x+h) - f(x)}{h} \qquad (4.1)$$

provided the limit exist. To find a finite difference approximation to $f'(x)$, we deleted the limit operation , then the result become the forward difference

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad (4.2)$$

when $x_{i+1} = x_{i+h} \Rightarrow \frac{f_{i+1} - f_i}{h} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$ (4.3)

where equation (4.3) is forward difference approximation of $f'(x_i)$
Assume that $f \in C^2$ in a neighborhood of $x = x_i$, then for h sufficiently small ,we have the Taylor expression

$$f_{i+1} = f_i + hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{3!}h^3 f_i''' \dots \dots \dots \qquad (4.4)$$

substitution into (4.3) we get

$$\frac{f_{i+1}-f_i}{h} = \frac{1}{h}\left[\left(f_i + +hf_i^{'} + \frac{1}{2}h^2 f_i^{''} + \frac{1}{3!}h^3 f_i^{'''} \ldots..\right) - f_i\right] = f_i^{'} + \frac{1}{2}hf_i^{''} + \frac{1}{3!}h^2 f_i^{'''}$$

So the leading error in (4.4) is $\frac{1}{2}hf_i^{''}$ so the approximation is first order in the step size h

4.1.Taylor Series:

By using a backward difference approximation to first derivatives, let $f \in C^2$ then

$$f_{i-1} = f_i - hf_i^{'} + \frac{1}{2}h^2 f_i^{''} + \cdots$$

It follow that $f_i^{'} = \frac{f_i - f_{i-1}}{h} + o(h)$

for higher order accuracy we drive Taylor expression for higher order from linear combination

$$f_{i+1} = f_i + hf_i^{'} + \frac{1}{2}h^2 f_i^{''} + \frac{1}{6}h^3 f_i^{'''} + \frac{1}{24}h^4 f_i^{''''} + \frac{1}{120}h^5 f^{'''''} + \cdots \qquad (4.5)$$

and $f_{i-1} = f_i - f_i^{'}h + \frac{h^2 f_i^{''}}{2} - \frac{h^3 f_i^{'''}}{3!} + \frac{h^4 f_i^{''''}}{4!} - \frac{h^5 f_i^{'''''}}{5!}$ (4.6)

by subtracting(**) from (*) we get

$$f_{i+1} - f_{i-1} = 2hf_i^{'}\frac{1}{3}h^3 f_i^{'''} + \cdots$$

on division by $2h$ gives

$$f_i^{'} = \frac{f_{i+1}-f_{i-1}}{2h} - \frac{h^2 f^{'''}}{6} - \frac{h^4 f^{'''''}}{120} \ldots \ldots \ldots.$$

4.2.Explicit(Forward) Euler Method:

Consider the equation

$$u^{'} = f(u,t) \qquad (4.7)$$

with $u(t_0) = u_0$

by replacing $u^{'}$ with a first forward difference and evaluate f at time t_0,we obtain

$$f\,(u_n, t_n) \approx \frac{u_{n+1}-u_n}{h} \text{o} u_{n+1} \approx u_n + hf(u_n, t_n\text{r}\,) \qquad (4.8)$$

So if u_n is known , we can evaluate the right- hand side and explicitly calculate u_{n+1}. The grid function values at new time step can be directly evaluate from values at previous time steps without numerical linear algebra or iteration, so Euler's method is an explicit single step method. To investigate the truncation error for Euler's method for first order approximation for sufficiently small h

Let $\frac{u_{n+1}-u_n}{h} + o(h) = f(u_n, t_n)$ (4.9)

so if $h \to 0$, we get the original differential equation

Thus Euler method is said to be consistent with the differential equation. Analogous to (4.2) we have

$$u_{n+1} = u_n + hf(u_n, t_n) + o(h^2) \qquad (4.10)$$

in equation (4.10) Euler method would appear to be second order accurate, the error h^2 is of order second this is called Local Truncation error.

4.3.Algorithm for Euler method

1) Define the function $f(t,y)$, such that $f(t,y) \in [a,b]$
2) Give initial value for t_o and y_o
3)Specify the step size $h = \frac{b-a}{n}$, where number of steps
4) out put t_o and y_o
5) for j from 1to n do
6) Let $k_j = f(t_j, y_j)$, $y_{j+1} = y_j + h * k_j$, and $t_{j+1} = t_j + h$
7)output t_j and y_j
8) End.

4.4.Modified Euler Method

Let $p(x_0, y_0)$ be the point on the solution curve.

Let PA be the tangent at (x_0, y_0) to the curve. Now let the this tangent the ordinate at $x = x_0 + \frac{h}{2}$ at N_1 and in

y −coordinate of $N_1 = y_0 + \frac{h}{2}f(x_0, y_0)$ (4.11)

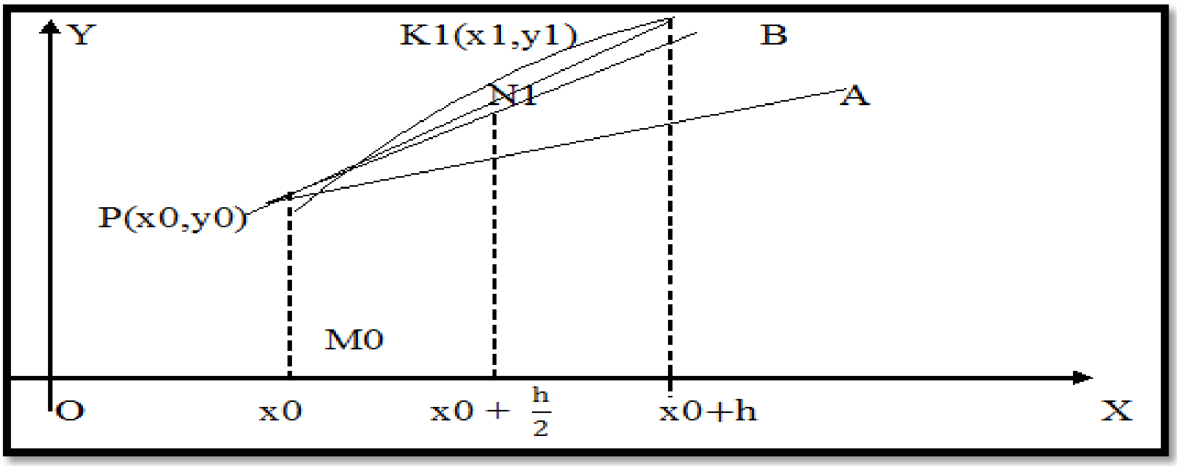
The slope at $N_1$is $f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0))$. So by using approximation

$x = x_0 + h = x_1$ and $y_1{}^{(1)}$is taken as approximate value of y.

Then $y_1{}^{(1)} = y_0 + h[f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0))]$. (4.12)

for n iteration $y_{n+1} = y_n + h[f(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n))]$. (4.13)

Equations (4.2) or (4.3) is called modified Euler's formula.



(4.1): The Figure Show the descretization plane and the tangent slope.

4.5.Algorithm for Modified Euler Method.

From Euler Method let

$k_1 = f(t_0, y_0)$

$k_2 = f(t_0 + \frac{h}{2}; y_0 + \frac{h}{2} * k_1)$

$y_1 = y_0 + h * k_2$

The general step is obtained by replacing t_0, y_0 and y_1 by t_j , y_j and y_{j+1} respectively

$y_{j+1} = y_j + h * (f(t_j + \frac{h}{2}; y_j + \frac{h}{2} * f(t_j, y_j))$

V. METHODS IMPLANTATIONS

Let us consider the following initial boundary value problem $y' = 2xy, y(0) = 1$.
in this case we compare between analytical and numerical solution using discretization, techniques and calculate the local truncation errors. The results are displayed in the following tables.

I:The Table shows Analytical and Corresponding Euler Numerical Method Solution of $y' = 2xy, y(0) = 1$ for different step size h and t .

t	h= 0.05	h=0.025	h=0.01	h=0.001	Exact
0.0	1.00000000000	1.0000000000	1.000000000	1.000000000	1.000000000
0.1	1.01000000000	1.0050000000	1.0020000000	1.000200000	1.0100501671
0.2	1.03020000000	1.0150500000	1.0060080000	1.000600080	1.0408107742
0.3	1.06110600000	1.0252050000	1.0120440480	1.001200440	1.0941742837
0.4	1.10355024000	1.0302757500	1.0201403974	1.0020014004	1.173510871
0.5	1.15872775200	1.0560326438	1.0303418014	1.003034018	1.2840254167
0.6	1,22825141710	1.0877136231	1.0427068175	1.0042070059	1.433294146
0.7	1.31422901630	1.2578359990	1.0573047130	1.0056128957	1.632316220
0.8	1.41936733761	1.1708149439	1.0742215884	1.0072218763	1.8964808793
0.9	1.54711039800	1.2235016164	1.0935575770	1.0090348757	2.2479079867
1.0	1.70182143780	1.2846766972	1.1154287285	1.011529455	2.7182818285

II: The Table shows Analytical and Corresponding Euler Modified Numerical Method Solution of $y' = 2xy, y(0) = 1$ for different step size h and t .

t	h= 0.05	h=0.025	h=0.01	h=0.001	Exact
0.0	1.0025000000	1.0006250000	1.000100000	1.000001000	1.00000000
0.1	1.10055137813	1.0062575869	1.0022021002	1.0002020203	1.0100501671
0.2	1.13063658200	1.0170126380	1.0042603523	1.0006031815	1.0408107742
0.3	1.16793345620	1.0330226416	1.0104047184	1.0012047244	1.0941742837
0.4	1.21856337150	1.0545417945	1.0186217337	1.0020070102	1.173510871
0.5	1.28413731290	1.0819022109	1.0289646672	1.0025087655	1.2840254167
0.6	1.36559977370	1.1155324649	1.311178098	1.0037135009	1.433294146
0.7	1.46807096670	1.1559687737	1.0457583422	1.005121675	1.632316220
0.8	1.59403145560	1.2030239276	1.0627297452	1.0067321622	1.8964808793
0.9	1.74811452120	1.2591468735	1.0821382723	1.0085469186	2.2479079867
1.0	1.93625534650	1.3244847917	1.1041067614	1.0105670391	2.7182818285

III: The Table shows The Percentages Local Truncation Errors of $y' = 2xy, y(0) = 1$ for different step size h and t Using Euler Method.

t=ti	h= 0.05	h=0.025	h=0.01	h=0.001
0.0	0.00%	0.00%	0.00%	0.00%
0.1	0.005%	0.500%	0.797%	0.975%
0.2	1.061%	2.475%	3.344%	3.863%
0.3	3.022%	6.303%	7.506%	8.497%
0.4	5.961%	12.206%	13.069%	14.615%
0.5	9.758%	17.756%	19.757%	21.883%
0.6	14.306%	24.111%	27.251%	29.937%
0.7	19.487%	29.068%	35.228%	38.372%
0.8	25.158%	38.264%	43.357%	46.890%
0.9	31.175%	45.571%	51.352%	55.112%
1.0	37.393%	52.739%	58.966%	62.788%

IV: The Table shows The Percentages Local Truncation Errors of $y' = 2xy, y(0) = 1$ for different step size h and t Using Modified Euler Method.

t	h= 0.05	h=0.025	h=0.01	h=0.001
0.0	-0.250%	-0.063%	-0.01%	-0.0001%
0.1	-8.960%	-8.960%	0.777%	0.975%
0.2	-8.630%	-8.630%	3.512%	3.861%
0.3	-6.741%	-6.686%	7.660%	8.496%
0.4	-2.629%	-3.834%	13.199%	14.614%
0.5	-.087%	15.741%	19.864%	21.925%
0.6	4.722%	22.184%	28.059%	30.056%
0.7	10.062%	29.182%	35.934%	38.424%
0.8	15.948%	36.565%	43.963%	46.916%
0.9	22.233%	43.986%	51.860%	55.134%
1.0	28.769%	51.275%	59.382%	62.282%

VI. RESULT DISCUSSION

In this study, the accuracy of first order ordinary differential equation has been tested using Euler and modified Euler algorithm. The comparison were made between analytical and numerical solutions in order to derive percentages local truncation errors. The tables observations clearly that the Euler and Euler modified methods results are not enough accurate for first order differential equations except when h and t are very small. Generally the modified Euler method is more accurate than Euler method.

VII. CONCLUSION

In this work which concern with the accuracy of numerical solutions for first order differential equations. Euler and modified Euler methods have been applied in order to investigate the objective of the study. It found that the use of Euler and Euler modified in solving first order differential equations are not more accurate except for very small size of h and t . Also the local truncation errors between Euler and modified Euler methods are very closed, although there is discrepancy between accuracy analytical and numerical results.

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