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Some Fixed Point and Common Fixed Point Theorems for Rational Inequality in Hilbert Space

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Abstract :- *There are several theorems are prove in Hilbert space, using various type of mappings . In this paper, we prove some fixed point theorem and common fixed point . Theorems, in Hilbert space using different, symmetric rational mappings . The object of this paper is to obtain a common unique fixed point theorem for four continuous mappings defined on a non-empty closed subset of a Hilbert space .*

Keywords :- *Fixed point , Common Fixed point , Hilbert space , rational inequality , continuous mapping .*

I. Introduction

The study of properties and applications of fixed points of various types of contractive mapping in Hilbert and Banach spaces were obtained among others by Browder [1] ,Browder and Petryshyn [2,3] , Hicks and Huffman [5] ,Huffman [6] , Koparde and Waghmode [7] . In this paper we present some fixed point and common fixed point theorems for rational inequality involving self mappings . For the purpose of obtaining the fixed point of the four continuous mappings . we have constructed a sequence and have shown its convergence to the fixed point .

II. MAIN RESULTS

Theorem 1 :-

Let E, F and T be for continuous self mappings of a closed subset C of a Hilbert space H satisfying conditions :

$1c_1$: $E(H) \subset T(H)$ and $F(H) \subset T(H)$

$ET = TE$, $FT = TF$

$1c_2$:

$$\|Ex - Fy\| \le \alpha \left[\frac{\|Tx - Ty\|\{\|Tx - Ex\| + \|Ty - Fy\|\}}{\|Tx - Fy\| + \|Ty - Ex\|} \right] + \beta\left[\|Tx - Ex\| + \|Ty - Fy\|\right] + \gamma\left[\|Tx - Fy\| + \|Ty - Ex\|\right] + \delta\|Tx - Ty\|$$

For all $x, y \in C$ with $Tx \ne Ty$, where non negative $\alpha, \beta, \gamma, \delta$ such that $0 \le \alpha + \beta + \gamma + \delta < 1$. Then E, F, T have unique common fixed point .

Proof :- Let $x_0 \in C$, Since $E(H) \subset T(H)$ we can choose a point $x_1 \in C$, such that $Tx_1 = Ex_0$, also $F(H) \subset T(H)$, we can choose $x_2 \in C$ such that In general we can choose the point :

$$Tx_{2n+1} = Ex_{2n} \qquad \text{............(1.1)}$$

$$Tx_{2n+2} = Fx_{2n+2} \qquad \text{............(1.2)}$$

Now consider

$$\left\|Tx_{2n+1} - Tx_{2n+2}\right\| = \left\|Ex_{2n} - Fx_{2n+1}\right\|$$

From $1c_2$

$$\left\|Tx_{2n} - Fx_{2n+1}\right\| \leq \alpha \left[\frac{\left\|Tx_{2n} - Tx_{2n+1}\right\|\left\{\left\|Tx_{2n} - Ex_{2n}\right\| + \left\|Tx_{2n+1} - Fx_{2n+1}\right\|\right\}}{\left\|Tx_{2n} - Fx_{2n+1}\right\| + \left\|Tx_{2n+1} - Ex_{2n}\right\|} \right]$$

$$+ \beta \left[\left\|Tx_{2n} - Ex_{2n}\right\| + \left\|Tx_{2n+1} - Fx_{2n+1}\right\|\right]$$

$$+ \gamma \left[\left\|Tx_{2n} - Fx_{2n+1}\right\| + \left\|Tx_{2n+1} - Ex_{2n}\right\|\right] + \delta \left\|Tx_{2n} - Tx_{2n+1}\right\|$$

$$\left\|Tx_{2n+1} - Fx_{2n+2}\right\| \leq \alpha \left[\frac{\left\|Tx_{2n} - Tx_{2n+1}\right\|\left\{\left\|Tx_{2n} - Tx_{2n+1}\right\| + \left\|Tx_{2n+1} - Tx_{2n+2}\right\|\right\}}{\left\|Tx_{2n} - Tx_{2n+2}\right\| + \left\|Tx_{2n+1} - Tx_{2n}\right\|} \right]$$

$$+ \beta \left[\left\|Tx_{2n} - Tx_{2n+1}\right\| + \left\|Tx_{2n+1} - Tx_{2n+2}\right\|\right]$$

$$+ \gamma \left[\left\|Tx_{2n} - Tx_{2n+2}\right\| + \left\|Tx_{2n+1} - Tx_{2n+1}\right\|\right] + \delta \left\|Tx_{2n} - Tx_{2n+1}\right\|$$

$$\left\|Tx_{2n+1} - Tx_{2n+1}\right\| \leq \left[\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma} \right] \left\|Tx_{2n} - Tx_{2n+1}\right\|$$

$$\left\|Tx_{2n+1} - Tx_{2n+1}\right\| \leq q \left\|Tx_{2n} - Tx_{2n+1}\right\|$$

where

$$q = \left[\frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma} \right] < 1;$$

For n= 1,2,3, …….

Whether , $\left\|Tx_{2n+1} - Tx_{2n+2}\right\| = 0$ or not

Similarly , we have

$$\left\|Tx_{2n+1} - Tx_{2n+2}\right\| \leq q^n . \left\|Tx_0 - Tx_1\right\|$$

For every positive integer n , this means that ,

$$\sum_{i=0}^{\infty} \left\|Tx_{2i+1} - Tx_{2i+2}\right\| \leq \infty$$

The sequence $\left(T^n x_0\right)_{n \in N}$ converges to same u in C , so by (1.1) & (1.2) :

$\left\{E^n x_0\right\}_{n \in N}$ and $\left\{F^n x_0\right\}_{n \in N}$ also converges to the some point u ,respectively.

Since E, F, T are continuous , there is a subsequence t of $\left\{F^n x_0\right\}_{n \in N}$ such that :

$$E\left[T(t)\right] \to E(u), T\left[E(t)\right] \to T(u), F\left[T(t)\right] \to F(u), T\left[F(t)\right] \to T(u)$$

By $(1c_1)$ we have , $E(u) = F(u) = T(u)$ ……………(1.3)

Thus , we can write

$T(Tu) = T(Eu) = E(Tu) = E(Eu) = E(Fu) = T(Fu) = F(Tu) = F(Eu) = F(Fu)$ ………(1.4)

By $(1c_2)$, (1.3) and (1.4) we have , if $E(u) \neq F(Eu)$

$$\|Eu - F(Eu)\| \leq \alpha \left[\frac{\|Tu - T(Eu)\| \left[\|Tu - Eu\| + \|T(Eu) - F(Eu)\| \right]}{\|Tu - F(Eu)\| + \|T(Eu) - Eu\|} \right]$$

$$+ \beta \left[\|Tu - Eu\| + \|T(Eu) - F(Eu)\| \right]$$

$$+ \gamma \left[\|Tu - F(Eu)\| + \|T(Eu) - Eu\| \right] + \delta \|Tu - T(Eu)\|$$

$$\|Eu - F(Eu)\| \leq (\beta + \gamma + \delta) \|Eu - F(Eu)\|$$

Thus we get a contraction ,

Hence $Eu = F(Eu)$ ………………(1.5)

From (1.4) and (1.5) we have

$$Eu = F(Eu) = T(Eu) = E(Eu)$$

Hence Eu is a common fixed point of E, F and T .

Uniqueness :-

Let v is another fixed point of E, F and T different u ,

Then by $(1c_2)$ we have

$$\|u - v\| = \|Eu - Fv\|$$

$$\|Eu - Fv\| \leq \alpha \left[\frac{\|Tu - Tv\| \left[\|Tu - Eu\| + \|Tv - Fv\| \right]}{\|Tu - Fv\| + \|Tv - Eu\|} \right]$$

$$+ \beta \left[\|Tu - Eu\| + \|Tv - Fv\| \right] + \gamma \left[\|Tu - Fv\| + \|Tv - Eu\| \right] + \delta \|Tu - Tv\|$$

$$\|u - v\| \leq (2\gamma + \delta) \|u - v\|$$

Which is a contradiction .

Therefore u is a unique fixed point of E, F & T in C .

Hence Proved

Theorem 2 :-

Let E, F and T be for continuous self mappings of a closed subset C of a Hilbert space H satisfying the following condition :

$2c_1$: $E(H) \subset T(H)$ & $F(H) \subset T(H)$

$ET = TE$, $FT = TF$

$2c_2$: $$\|E^r x - F^s y\| \leq \alpha \left[\frac{\|Tx - Ty\| \left[\|Tx - E^r x\| + \|Ty - Fy\| \right]}{\|Tx - F^s y\| + \|Ty - E^r x\|} \right]$$

$$+ \beta \left[\|Tx - E^r x\| + \|Ty - F^s y\| \right] + \gamma \left[\|Tx - F^s y\| + \|Ty - E^r x\| \right]$$

$$+ \delta \|Tx - Ty\|$$

For all x, y in C , where non negative $\alpha, \beta, \gamma, \delta$ such that

$0 \leq \alpha + \beta + \gamma + \delta < 1$ with $Tx \neq Ty$, If some positive integers r, s exixts

Such that E^r, F^s & T are continuous .

Then E, F, T have unique common fixed point.

Proof :- we have

$$E(H) \subset T(H) \quad \& \quad F(H) \subset T(H)$$

$$ET = TE \quad , \quad FT = TF$$

It follows that :

$$E^{r}(H) \subset T(H) \quad \& \quad F^{s}(H) \subset T(H)$$

$$E^{r}T = TE^{r} \quad , \quad F^{s}T = TF^{s}$$

By theorem (1) , there is a unique fixed point in C such that ,

$$u = Tu = E^{r}u = F^{s}u \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

i.e u is the unique fixed point of T , E^{r} & F^{s} .

$$Now \qquad T(Eu) = E(Tu) = Eu = E\left(E^{r}u\right) = E^{r}(Eu) \qquad \ldots\ldots(2.2)$$

$$and \qquad T(Fu) = F(Tu) = Fu = F\left(F^{s}u\right) = F^{s}(Fu) \qquad \ldots\ldots\ldots(2.3)$$

Hence it follows that Eu is a common fixed point of E^{r} & T , similarly Fu is a common fixed point of T & F^{s} in X . The uniqueness of u from
(2.1) , (2.2) &(2.3),
Implies that : $u = Eu = Fu = Tu$
This complete the proof of the theorem .
Remark :-

(i) If $r = s = 1$ the we get theorem 1 .

……………….x…………………..

Theorem 3 :-
Let A, B, S & T be continuous self mappings of a closed subset C of a Hilbert Space H satisfying the following condition :

$3c_1$: $A(H) \subseteq T(H)$ & $B(H) \subseteq T(H)$

$$AS = SA \quad , \quad BT = TB$$

$3c_2$: $\|Ax - By\| \le \alpha \|Sx - Ty\| + \beta_{\max}\left[\|Sx - Ax\|, \|Ty - By\|, \|Sx - By\|, \|Ty - Ax\|\right]$

For all x, y in C with $Tx \ne Ty$, where non negative such that $0 \le \alpha + \beta < 1$;
Then A, B, S, T have unique common fixed point in C .

Proof :- Let x_0 be an arbitrary point of C , since $A(H) \subseteq T(H)$ we can choose the point x_1 & y_0 in C such that ,

$$Ax_0 = Tx_1 = y_0$$

Also $B(H) \subseteq S(H)$,we can choose the point x_2 & y_1 in C such that

$$Bx_1 = Sx_2 = y_1$$

In general we can choose the points

$$Tx_{2n+1} = Ax_{2n} = y_{2n} \qquad \ldots\ldots\ldots\ldots(3.1)$$

$$and \qquad Sx_{2n+2} = Bx_{2n+1} = y_{2n+1} \qquad \ldots\ldots\ldots\ldots(3.2)$$

For all $n = 0, 1, 2, 3, \ldots\ldots.$
Now consider ,

$$\|y_{2n} - y_{2n+1}\| = \|Ax_{2n} - Bx_{2n+1}\|$$

From $3c_2$:

$$\|Ax_{2n} - Bx_{2n+1}\| \le \alpha \|Sx_{2n} - Tx_{2n+1}\| + \beta_{\max}\left[\|Sx_{2n} - Ax_{2n}\|, \|Tx_{2n+1} - Bx_{2n+1}\|, \|Sx_{2n} - Bx_{2n+1}\|, \|Tx_{2n+1} - Ax_{2n}\|\right]$$

$$\|y_{n} - y_{2n+1}\| \le \alpha \|y_{2n-1} - y_{2n}\| + \beta_{\max}\left[\|y_{2n} - y_{2n-1}\|, \|y_{2n+1} - y_{2n-1}\|, \|y_{2n} - y_{2n+1}\|\right] \qquad \ldots\ldots\ldots(3.3)$$

There arise three cases ,

Case 1:- If we take max is $\|y_{2n-1} - y_{2n}\|$, then (3.3) gives ,

$$\|y_{2n+1} - y_{2n}\| \le (\alpha + \beta)\|y_{2n-1} - y_{2n}\|$$

Case 2 :- If we take max is $\|y_{2n+1} - y_{2n}\|$, then (3.3) gives ,

$$\|y_{2n+1} - y_{2n}\| \le \frac{\alpha}{1-\beta}\|y_{2n-1} - y_{2n}\|$$

Case 3 :- If we take max is $\|y_{2n+1} - y_{2n-1}\|$, then (3.3) gives ,

$$\|y_{2n+1} - y_{2n}\| \le \frac{\alpha+\beta}{1-\beta}\|y_{2n-1} - y_{2n}\|$$

From the above cases 1,2 ,3 we observe that ,

$$\|y_{2n+1} - y_{2n}\| \le q\,\|y_{2n-1} - y_{2n}\|$$

$$where \qquad q = \max\left[(\alpha+\beta), \frac{\alpha}{1-\beta}, \frac{\alpha+\beta}{1-\beta}\right] < 1$$

$for \quad n = 1, 2, 3, \ldots\ldots\ldots.$

Similarly we have ,

$$\|y_{2n+1} - y_{2n}\| \le q^n\|y_0 - y_1\|$$

For every positive integer n , this means that ,

$$\sum_{i=0}^{\infty}\|y_{2i+1} - y_{2i}\| < \infty$$

Thus the completeness of the space implies that the sequence $\{y_n\}_{n\in N}$ converges to the some point u in C , so by (3.1) & (3.2) the sequence

$\{A^n x_0\}, \{B^n x_0\}, \{S^n x_0\}, \{T^n x_0\}$ also converges to the some points u respectively ;

Since A, B, S, T are continuous , this implies

$$Tx_{2n+1} = Ax_{2n} = y_{2n} \to u \quad \text{as} \quad n \to \infty$$

$$Sx_{2n+2} = Bx_{2n+1} = y_{2n+1} \to u \quad \text{as} \quad n \to \infty$$

The pair (A, S) and (B, T) are weakly compatible which gives that , u is a common fixed point of A , B , S & T.

Uniqueness :-

Let as assume that w is another fixed point of A , B , S & T different from u , i.e. $u \ne w$ then

$$Tu = Au = u \qquad \& \qquad Sw = Bw = w$$

From $3c_2$ we have ,

$$\|u - w\| < (\alpha + \beta)\|u - w\|$$

which contradiction .

Hence u is a unique common fixed point of A , B , S ,T in C .

This complete the proof of the theorem.

…………..x……………

Theorem 4 :-

Let E , F & T be for continuous self mappings of a closed subset C of a Hilbert Space H satisfying the following condition

$4c_1$:
$$E(H) \subset T(H) \quad \& \quad F(H) \subset T(H)$$
$$ET = TE \quad , \quad FT = TF$$

$4c_2$:
$$\{\|Ex - Fy\|\}^2 \le \alpha\,\|Tx - Ex\|\|Ty - Fy\| + \beta\,\|Tx - Fy\|\|Ty - Ex\|$$
$$+\gamma\,\|Tx - Ex\|\|Ex - Ty\| + \delta\,\|Tx - Ty\|\|Ty - Fy\|$$

For all $x, y \ in \ C$, where non negative $\alpha, \beta, \gamma, \delta$ such that $0 \le \alpha + \beta + \gamma + \delta < 1$, with $Tx \neq Ty$ then E , F , T have unique common fixed point .

PROOF :-

Let $x_0 \in C$, Since $E(H) \subset T(H)$ we can choose a point $x_1 \in C$,

Such that $Tx_1 = Ex_0$, also $F(H) \subset T(H)$, we can choose $x_2 \in C$ such that

$Tx_2 = Fx_1$.

In general we can choose the point :

$$Tx_{2n+1} = Ex_{2n} \qquad \ldots\ldots\ldots.(4.1)$$

$$Tx_{2n+2} = Fx_{2n+1} \qquad \ldots\ldots\ldots..(4.2)$$

$for \ every \quad n \in N$

We have

$$\left\|Tx_{2n+1} - Tx_{2n+2}\right\|^2 = \left\|Ex_{2n} - Fx_{2n+1}\right\|^2$$

$$\begin{aligned}\left\|Ex_{2n} - Fx_{2n+1}\right\|^2 \le\ & \alpha \left\|Tx_{2n} - Ex_{2n}\right\|\left\|Tx_{2n+1} - Fx_{2n+1}\right\| \\ & + \beta \left\|Tx_{2n} - Fx_{2n}\right\|\left\|Tx_{2n+1} - Ex_{2n+1}\right\| \\ & + \gamma \left\|Tx_{2n} - Ex_{2n}\right\|\left\|Ex_{2n} - Tx_{2n+1}\right\| \\ & + \delta \left\|Tx_{2n} - Tx_{2n+1}\right\|\left\|Tx_{2n+1} - Fx_{2n+1}\right\|\end{aligned}$$

$$\begin{aligned}\left\|Tx_{2n+1} - Tx_{2n+2}\right\|^2 \le\ & \alpha \left\|Tx_{2n} - Tx_{2n+1}\right\|\left\|Tx_{2n+1} - Tx_{2n+2}\right\| \\ & + \beta \left\|Tx_{2n} - Tx_{2n+2}\right\|\left\|Tx_{2n+1} - Tx_{2n+1}\right\| \\ & + \gamma \left\|Tx_{2n} - Tx_{2n+1}\right\|\left\|Tx_{2n+1} - Tx_{2n+1}\right\| \\ & + \delta \left\|Tx_{2n} - Tx_{2n+1}\right\|\left\|Tx_{2n+1} - Tx_{2n+2}\right\|\end{aligned}$$

$$\left\|Tx_{2n+1} - Tx_{2n+2}\right\| \le (\alpha + \delta)\left\|Tx_{2n} - Tx_{2n+1}\right\|$$

For n = 1,2,3,……..

Whether $\left\|Tx_{2n+1} - Tx_{2n+2}\right\| = 0$ or not

Similarly we have

$$\left\|Tx_{2n+1} - Tx_{2n+2}\right\| \le (\alpha + \delta)^n \left\|Tx_0 - Tx_1\right\|$$

For every positive integer n , this means that ,

$$\sum_{i=0}^{\infty} \left\|Tx_{2i+1} - Tx_{2i+2}\right\| < \infty$$

The sequence $\{T^n x_0\}_{n \in N}$ converges to some u by (4.1) & (4.2).

$\{E^n x_0\}_{n \in N}$ and $\{F^n x_0\}_{n \in N}$ also converges to the some point respectively.

Since E , F , T are continuous ,this is a subsequence t of $\{T^n x_0\}_{n \in N}$ such that ,

$$E[T(t)] \to E(u) \quad , \quad T[E(t)] \to T(u)$$

$$F[T(t)] \to F(u) \quad , \quad T[F(t)] \to T(u)$$

By ($4c_1$) we have ,

$E(u) = F(u) = T(u)$ ....(4.3)

thus, $T(Tu) = T(Eu) = E(Tu) = E(Eu) = E(Fu) = T(Fu) = F(Tu) = F(Eu) = F(Fu)$ ....(4.4)

by $4c_2$, (4.3) & (4.4) we have

$$E(u) \neq F(Eu)$$

$$\|Eu - F(Eu)\|^2 \leq \alpha \|Tu - Eu\| \|T(Eu) - F(Eu)\|$$
$$+ \beta \|Tu - F(Eu)\| \|T(Eu) - Eu\| + \gamma \|Tu - Eu\| \|Eu - T(Eu)\|$$
$$+ \delta \|Tu - T(Eu)\| \|Tu - F(Eu)\|$$

$$\|Eu - F(Eu)\| \leq 0$$

thus we get a contradiction .

Hence $Eu = F(Eu)$ …………..(4.5)

From (4.4) & (4.5) we have

$$Eu = F(Eu) = T(Eu) = E(Eu)$$

Hence Eu is a common fixed point of E , F & T .

Uniqueness :-

Let v is another fixed point of E , F & T different from then by $1c_2$ we have ,

$$\|u - v\|^2 = \|Eu - Fv\|^2$$

$$\|Eu - Fv\|^2 \leq \alpha \|Tu - Eu\| \|Tv - Fv\| + \beta \|Tu - Fv\| \|Tv - Eu\|$$
$$+ \gamma \|Tu - Eu\| \|Eu - Tv\| + \delta \|Tu - Tv\| \|Tv - Fv\|$$

$$\|u - v\| \leq \beta \|u - v\|,$$

Which is a contradiction.

Therefore u is unique fixed point of E , F & T .

Hence Proved

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