American Journal of Engineering Research (AJER) 2015
American Journal of Engineering Research (AJER)
e-ISSN: 2320-0847 p-ISSN : 2320-0936
Volume-4, Issue-10, pp-127-133
www.ajer.org
Research Paper Open Access

Solution of the Linear and Non-linear Partial Differential Equations Using Homotopy Perturbation Method

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Abstract:- *In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical, and engineering problems linear and nonlinear. This paper applies the homotopy perturbation method (HPM) to find exact solution of partial differential equation with the Dirichlet and Neumann boundary conditions.*

Keywords:- *homotopy perturbation method, wave equation, Burgers equation, homogeneous KdV equation.*

I. INRODUCTION

The notion of homotopy is an important part of topology and thus of differential geometry. The homotopy continuation method or shortly speaking homotopy was known as early as in the 1930s. Thus, in 1892, Lyapunov [1] introduced the so called "artificial small parameters method" considering a linear differential equation with variable coefficient in the form $\frac{du}{dt} = M(t)\,u$

with $M(t)$ a time periodic matrix. He replaced this equation with the equation $\frac{du}{dt} = \varepsilon M(t)\,u$.

To get the solution of the last equation, Lyapunov developed the power series over ε for the variable u and then setting $\varepsilon = 1$.

Later, this method was used by kinematicians in the 1960s in the US for solving mechanism synthesis problems [29]. The latest development was done by Morgan at General Motors [3]. There are also two important literature studies by Garcia and Zangwill [5] and Allgower and Georg [8]. The HPM was introduced by Ji-Huan He of Shanghai University in 1998, [9-13]. The HPM is a special case of the homotopy analysis method (HAM) developed by Liao Shijunin 1992 [25]. HPM has been applied by many authors, to solve many types of the linear and nonlinear equations in science and engineering, boundary value problems [2,11], Cauchy reaction–diffusion problem [4], heat transfer[6],nonlinear wave equations [9], non-linear oscillators with discontinuities [12], Sumudu transform[21], and to other fields [13-28].

The method employs a homotopy transform to generate a convergent series solution of linear and nonlinear partial differential equations. The homotopy perturbation method is combination of perturbation and homotopy method

II. HOMOTOPY PERTURBATION METHOD

To illustrate the basic idea of this method, we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (1)$$

With the following boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \quad (2)$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω . The operator A can be decomposed into a linear and a non-linear, designated as L and N respectively. The equation (1)can be written as the following form.

$$L(u)+N(u)-f(r)=0 \quad (3)$$

Using homotopy perturbation technique, we construct a homotopy $v(r,p):\Omega\times[0,1]\to \boldsymbol{R}$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0 \quad (4)$$

Where $p\in(0,1)$ is an embedding parameter, u_0 is an initial approximation solution of (1), which satisfies the boundary, form equation (4) we obtain

$$H(v,0)=L(v)-L(u_0)=0 \quad (5)$$

$$H(v,1)=A(v)-f(r)=0 \quad (6)$$

Changing the process of p from zero to unity, a change $v(r,p)$ from $u_0(r)$ to $u(r)$.

In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the Solutions of equation (4) can be written as a power series in p as the following

$$v=v_0+pv_1+p^2v_2+p^3v_3+\cdots \quad (7)$$

Setting $p=1$ results in the approximate of equation (7), can be obtained

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+v_3+\cdots . \quad (8)$$

III. Applications

In this section, we apply Homotopy perturbation method for solving linear and nonlinear problems.

Example 1.

Use the Homotopy perturbation method to solve the Laplace equations:

$$\begin{cases} u_{xx}+u_{yy}=0, & 0<x,\ y<\pi \\ u(0,y)=0, & u(\pi,y)=\sinh\pi\ \sin y \\ u(x,0)=0, & u(x,\pi)=0 \end{cases} \quad (9)$$

Using HPM, we construct a homotopy in the following form

$$H(v,p)=(1-p)\left[\frac{\partial^2 v}{\partial y^2}-\frac{\partial^2 u_0}{\partial y^2}\right]+p\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]=0 \quad (10)$$

We select $u_0(x,y)=y\sinh x$ as in initial approximation that satisfies the two conditions. Substituting equation (7) into equation (10) end equating the terms with identical powers of p, we drive

$$p^0:\ \begin{cases} \dfrac{\partial^2 v_0}{\partial y^2}-\dfrac{\partial^2 u_0}{\partial y^2}=0, \\ v_0(x,0)=0, \quad v_0(0,y)=0 \end{cases} \quad (11)$$

$$p^1:\ \begin{cases} \dfrac{\partial^2 v_1}{\partial y^2}+\dfrac{\partial^2 u_0}{\partial y^2}+\dfrac{\partial^2 v_0}{\partial x^2}=0, \\ v_1(x,0)=0, \quad v_1(0,y)=0 \end{cases} \quad (12)$$

$$p^2: \begin{cases} \dfrac{\partial^2 v_2}{\partial y^2} + \dfrac{\partial^2 v_1}{\partial x^2} = 0, \\ v_2(x,0) = 0, \quad v_2(0,y)=0 \end{cases} \qquad (13)$$

$$p^3: \begin{cases} \dfrac{\partial^2 v_3}{\partial y^2} + \dfrac{\partial^2 v_2}{\partial x^2} = 0, \\ v_2(x,0) = 0, \quad v_2(0,y)=0 \end{cases} \qquad (14)$$

$$\vdots$$

Consider $v_0 = u_0(x,y) = y \sinh x$. Form equations (12),(13) and (14), we have

$$v_1 = -\frac{y^3}{3!}\sinh x$$

$$v_2 = \frac{y^5}{5!}\sinh x$$

$$v_3 = -\frac{y^7}{7!}\sinh x$$

$$\vdots$$

Therefore , the solution of equation(9) when $p \to 1$ we will be as follows:

$$u(x,y) = \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots \right)\sinh x = \sinh x \ \sin y$$

Because the boundary conditions are Neumann boundary conditions, an arbitrary constant must be added. Therefore, the exact solution in will be as follows:

$u(x,y) = \sinh x \ \sin y + C$.

Example .2

Use the Homotopy perturbation method to solve the inhomogeneous wave equation

$$\begin{cases} u_{tt} - u_{xx} + 2 = 0, \quad 0<x<\pi, \ t>0 \\ u(0,t) = 0, \quad u(\pi,t)=\pi^2 \\ u(x,0) = x^2, \quad u_t(x,0)= \sin x \end{cases} \qquad (15)$$

Using HPM, we construct a homotopy in the following form

$$H(v,p) = (1-p)\left[\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right] + p\left[-\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} - 2\right] = 0 \quad (16)$$

We select $u_0(x,t) = x^2 + t\sin x$ as in initial approximation that satisfies the three conditions. Substituting equation (7) into equation (16) end equating the terms with identical powers of p , we drive

$$p^0: \begin{cases} \dfrac{\partial^2 v_0}{\partial t^2} - \dfrac{\partial^2 u_0}{\partial t^2} = 0, \ v_0(0,t)= 0 \\ v_0(x,0) = x^2, \quad v_0(\pi,t)=\pi^2 \end{cases} \qquad (17)$$

$$p^1: \begin{cases} \dfrac{\partial^2 v_1}{\partial t^2} + \dfrac{\partial^2 u_0}{\partial t^2} - \dfrac{\partial^2 v_0}{\partial x^2} + 2 = 0, \quad v_1(0,t) = 0 \\ v_1(x,0) = 0, \quad v_1(\pi,t) = 0 \end{cases} \qquad (18)$$

$$p^2: \begin{cases} \dfrac{\partial^2 v_2}{\partial t^2} - \dfrac{\partial^2 v_1}{\partial x^2} + 2 = 0, \quad v_2(0,t) = 0 \\ v_2(x,0) = 0, \quad v_2(\pi,t) = 0 \end{cases} \qquad (19)$$

$$p^3: \begin{cases} \dfrac{\partial^2 v_3}{\partial t^2} - \dfrac{\partial^2 v_2}{\partial x^2} + 2 = 0, \quad v_3(0,t) = 0 \\ v_3(x,0) = 0, \quad v_3(\pi,t) = 0 \end{cases} \qquad (20)$$

$$\vdots$$

Consider $v_0 = u_0(x,t) = x^2 + t \sin x$. Form equations (18),(19) and (20), we have

$$v_1 = -\frac{t^3}{3!}\sin x$$
$$v_2 = \frac{t^5}{5!}\sin x$$
$$v_3 = -\frac{t^7}{7!}\sin x$$
$$\vdots$$

Therefore , the solution of equation(15) when $p \to 1$ we will be as follows:

$$u(x,t) = x^2 + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \right)\sin x = x^2 + \sin t \sin x$$

Because the boundary conditions are Neumann boundary conditions, an arbitrary constant must be added. Therefore, the exact solution in will be as follows:

$$u(x,t) = x^2 + \sin t \sin x + C .$$

Example.3

Use the Homotopy perturbation method to solve the Burgers equation

$$u_t + u u_x - u_{xx} = 0, \quad u(x,0) = x \quad (21)$$

Using HPM, we construct a homotopy in the following form

$$H(v,p) = (1-p)\left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right] = 0 \qquad (22)$$

Substituting equation (7) into equation (22) end equating the terms with identical powers of p , we drive

$$p^0: \begin{cases} \dfrac{\partial v_0}{\partial t} - \dfrac{\partial u_0}{\partial t} = 0, \quad v_0(x,0) = x \end{cases} \quad (23)$$

$$p^1: \begin{cases} \dfrac{\partial v_1}{\partial t} + \dfrac{\partial u_0}{\partial t} + v_0\dfrac{\partial v_0}{\partial x} - \dfrac{\partial^2 v_0}{\partial x^2} = 0, \quad v_1(x,0) = 0 \end{cases} \quad (24)$$

$$p^2: \left\{ \frac{\partial v_2}{\partial t} + v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0, \quad v_2(x,0) = 0 \right. \qquad (25)$$

$$p^3: \left\{ \frac{\partial v_3}{\partial t} + v_0 \frac{\partial v_2}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_2}{\partial x^2} = 0, \quad v_3(x,0) = 0 \right. \qquad (26)$$

$\vdots$

Consider $v_0 = u_0(x,t) = x$, as a first approximation for solution that satisfies the initial conditions. Form equations (24),(25) and (26), we have

$$v_1 = -xt$$
$$v_2 = x\ t^2$$
$$v_3 = -xt^3$$
$$\vdots$$

Therefore , the solution of equation(21) when $p \to 1$ we will be as follows:

$$u(x,t) = x\left(1 - t + t^2 - t^3 + \cdots\right) = \frac{x}{1-t}, \quad |t| < 1$$

Because the boundary conditions are Neumann boundary conditions, an arbitrary constant must be added. Therefore, the exact solution in will be as follows:

$$u(x,t) = \frac{x}{1-t} + C, \quad |t| < 1.$$

Example.4
Use the Homotopy perturbation method to solve the homogeneous KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x,0) = 6x \qquad (27)$$

Using HPM, we construct a homotopy in the following form

$$H(v,p) = (1-p)\left[\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p\left[\frac{\partial v}{\partial t} - 6v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right] = 0 \qquad (28)$$

Substituting equation (7) into equation (28) end equating the terms with identical powers of p , we drive

$$p^0: \left\{ \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad v_0(x,0) = 6x \right. \qquad (29)$$

$$p^1: \left\{ \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - 6v_0 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_0}{\partial x^3} = 0, \quad v_1(x,0) = 0 \right. \qquad (30)$$

$$p^2: \left\{ \frac{\partial v_2}{\partial t} - 6v_0 \frac{\partial v_1}{\partial x} - 6v_1 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_1}{\partial x^3} = 0, \quad v_2(x,0) = 0 \right. \qquad (31)$$

$$p^3: \left\{ \frac{\partial v_3}{\partial t} - 6v_0 \frac{\partial v_2}{\partial x} - 6v_1 \frac{\partial v_1}{\partial x} - 6v_2 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_2}{\partial x^3} = 0, \quad v_3(x,0) = 0 \right. \qquad (32)$$

$\vdots$

Consider $v_0 = u_0(x,t) = 6x$, as a first approximation for solution that satisfies the initial conditions. Form equations (30),(31) and (32), we have

$$v_1 = 6^3 x t$$
$$v_2 = 6^5 x \; t^2$$
$$v_3 = 6^7 x \; t^3$$
$$\vdots$$

Therefore , the solution of equation(1) when $p \to 1$ we will be as follows:

$$u(x,\mathrm{t}) = 6x + 6^3 x \; t + 6^5 x \; t^2 + 6^7 x \; t^3 + \cdots = 6x\left(1 + 36t + (36\,t)^2 + (36t)^3 + \cdots\right) = \frac{6x}{1-36t}, \quad |36t| < 1$$

Because the boundary conditions are Neumann boundary conditions, an arbitrary constant must be added. Therefore , the exact solution in will be as follows:

$$u(x,\mathrm{t}) = \frac{6x}{1-36t} + C, \quad |36t| < 1.$$

IV. Conclusion

In this paper, linear and nonlinear partial differential equations are solved by using Homotopy Perturbation Method. Analytical solution obtained by this method is satisfactory same as the exact results to these models. The homotopy perturbation method is powerful and efficient technique to find the solution of linear and non-linear equations.

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