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Some Fixed Point and Common Fixed Point Theorems in 2-Metric Spaces

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Abstract : *In the present paper we prove some fixed point and common fixed point theorems in 2-Metric spaces for new rational expression . Which generalize the well known results.*

Keywords : *Fixed point , 2-Metric Space , Common fixed point , Metric space , Completeness .*

I. Introduction

The concept of 2-metric space is a natural generalization of the metric space . Initially , it has been investigated by S. Gahler [1,2] . The study was further enhanced by B.E. Rhoades [6] , Iseki [3] , Miczko and Palezewski [4] and Saha and Day [7] , Khan[5] . Moreover B.E. Rhoades and other introduced several properties of 2- metric spaces and proved some fixed point and common fixed point theorems for contractive and expansion mappings and also have found some interesting results in 2-metric space , where in each cases the idea of convergence of sum of a finite or infinite series of real constants plays a crucial role in the proof of fixed point theorems . In this same way, we prove a fixed point theorem and common fixed point theorems for the mapping satisfying different types of contractive conditions in 2-metric space.

II. Definitions and Preliminaries

Definition 2.1 :- Let X be a non empty set . A real valued function d on $X \times X \times X$ is said to be a 2-metric in X if

(i) To each pair of distinct points x, y in X . There exists a point $z \in X$ such that $d(x, y, z) \neq 0$

(ii) $d(x, y, z) = 0$, When at least of x, y, z are equal.

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$

(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$

When d is a 2-metric on X , then the ordered pair (X, d) is called 2- metric space.

2.2 A sequence $\{x_n\}$ in 2-metric space (X, d) is said to be convergent to an element $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$.

It follows that if the sequence $\{x_n\}$ converges to x then

$\lim_{n\to\infty} d(x_n, \mathrm{a}, \mathrm{b}) = d(x, a, b)$ for all $a, b \in X$

2.3 A sequence $\{x_n\}$ in 2-metric space X is a Cauchy sequence if

$$d\left(x_m, x_n, a\right)=0 \text{ as } m,n\to\infty \text{ for all } a\in X .$$

2.4 If a sequence is convergent in a 2-metric space then it is a Cauchy Sequence .

2.5 A 2-metric space (X,d) is said to be complete if every Cauchy Sequence in X is convergent.

Proposition 2.6 :-

If a sequence $\{x_n\}$ in 2-metric space converges to x then every subsequence of $\{x_n\}$ also converges to the same limit x .

Proposition 2.7

Limit of a sequence in a 2-metric space , if exits , is unique .

3 Main Results :

Theorem 3.1

Let T be a mapping of a 2-metric spaces into itself. If T satisfies the following conditions:

$T^2=I$, where I is identity mapping ---------------------- (1.1)

$$\begin{aligned}d\left(Tx-Ty,a\right)\geq \alpha &\frac{d\left(x-Tx,a\right)d\left(y-Ty,a\right)}{d\left(x-y,a\right)}\\ +\beta &\frac{d\left(y-Ty,a\right)d\left(y-Tx,a\right)d\left(x-Ty,a\right)+\left[d\left(x-y,a\right)\right]^3}{\left[d\left(x-y,a\right)\right]^2}\\ +\gamma &\left[\frac{d\left(x-Tx,a\right)+d\left(y-Ty,a\right)}{2}\right]\\ +\delta &\left[\frac{d\left(x-Ty,a\right)+d\left(y-Tx,a\right)}{2}\right]+\eta d\left(x-y,a\right)\end{aligned} \qquad \text{…………..(1.2)}$$

Where $x\neq y, a>0$ is real with $8\alpha+10\beta+4\gamma+2\delta+3\eta>4$.

Then T has unique fixed point.

Proof :- Suppose x is any point in 2-metric space X .

Taking $y=\frac{1}{2}\left(T+I\right)x$, $z=T\left(y\right)$

$$\begin{aligned}d\left(z-x,a\right)&=d\left(Ty-T^2x,a\right)=d\left(Ty-T\left(Tx\right),a\right)\\ &\geq \alpha\frac{d\left(y-Ty,a\right)d\left(Tx-T\left(Tx\right),a\right)}{d\left(y-Tx,a\right)}\\ &+\beta\frac{d\left(Tx-T\left(Tx\right),a\right)d\left(Tx-Ty,a\right)d\left(y-T\left(Tx\right),a\right)+\left[d\left(y-Tx,a\right)\right]^3}{\left[d\left(y-Tx,a\right)\right]^2}\\ &+\gamma\left[\frac{d\left(y-Ty,a\right)+d\left(Tx-T(Tx),a\right)}{2}\right]\\ &+\delta\left[\frac{d\left(y-T\left(Tx\right),a\right)+d\left(Tx-Ty,a\right)}{2}\right]\\ &+\eta\left[d\left(y-Tx,a\right)\right]\end{aligned}$$

$$\geq \alpha \frac{d(y-Ty,a)d(Tx-x,a)}{\frac{1}{2}d(x-Tx,a)}$$

$$+\beta \frac{d(Tx-x,a)\left[d(Tx-y,a)+d(y-Ty,a)\right]d(y-x,a)+\left[d(y-Tx,a)\right]^3}{\frac{1}{4}\left[d(x-Tx,a)\right]^2}$$

$$+\gamma\left[\frac{d(y-Ty,a)+d(Tx-x,a)}{2}\right]$$

$$+\delta\left[\frac{d(y-x,a)+d(Tx-y,a)+d(y-Ty,a)}{2}\right]$$

$$+\eta\left[d(y-Tx,a)\right]$$

$$\geq 2\alpha d(y-Ty,a)$$

$$+\beta \frac{d(Tx-x,a)\left[\frac{1}{2}d(x-Tx,a)+d(y-Ty,a)\right].\frac{1}{2}d(x-Tx,a)+\frac{1}{8}\left[d(x-Tx,a)\right]^3}{\frac{1}{4}\left[d(x-Tx,a)\right]^2}$$

$$+\gamma\left[\frac{d(y-Ty,a)+d(Tx-x,a)}{2}\right]$$

$$+\delta\left[\frac{\frac{1}{2}d(x-Tx,a)+\frac{1}{2}d(x-Tx,a)+d(y-Ty,a)}{2}\right]+\frac{\eta}{2}d(x-Tx,a)$$

$$\geq 2\alpha d(y-Ty,a)$$

$$+\frac{\beta}{2}\left\{4\left[\frac{1}{2}d(x-Tx,a)+d(y-Ty)\right]+\frac{\left[d(x-Tx,a)\right]^3}{\left[d(x-Tx,a)\right]^2}\right\}$$

$$+\gamma\left[\frac{d(y-Ty,a)+d(Tx-x,a)}{2}\right]$$

$$+\delta\left[\frac{d(x-Tx,a)+d(y-Ty,a)}{2}\right]$$

$$+\frac{\eta}{2}\left[d(x-Tx,a)\right]$$

$$\geq 2\alpha d(y-Ty,a)+\frac{\beta}{2}\left[2d(x-Tx,a)+4d(y-Ty,a)+d(x-Tx,a)\right]$$

$$+\frac{\gamma}{2}\left[d(y-Ty,a)+d(Tx-x,a)\right]+\frac{\delta}{2}\left[d(x-Tx,a)+d(y-Ty,a)\right]+\frac{\eta}{2}d(x-Tx,a)$$

$$\geq d(x-Tx,a)\left(\frac{3\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}\right)+\left(2\alpha+2\beta+\frac{\gamma}{2}+\frac{\delta}{2}\right)d(y-Ty,a)$$

$$\geq \frac{1}{2}d(x-Tx,a)(3\beta+\gamma+\delta+\eta)+\frac{1}{2}d(y-Ty,a)(4\alpha+4\beta+\gamma+\delta)$$

Now for

$$d(u-x,a)=d(2y-z-x,a)=d(Tx-Ty,a)$$

$$\geq \alpha\frac{d(x-Tx,a)d(y-Ty,a)}{d(x-y,a)}+\beta\frac{d(y-Ty,a)d(y-Tx,a)d(x-Ty,a)+\left[d(x-y,a)\right]^3}{\left[d(x-y,a)\right]^2}$$

$$+\gamma\left[\frac{d(x-Tx,a)+d(y-Ty,a)}{2}\right]+\delta\left[\frac{d(x-Ty,a)+d(y-Tx,a)}{2}\right]+\eta d(x-y,a)$$

$$\geq \alpha\frac{d(x-Tx,a)d(y-Ty,a)}{\frac{1}{2}d(x-Tx,a)}$$

$$+\beta\frac{d(y-Ty,a)\frac{1}{2}d(x-Tx,a)\left[\frac{1}{2}d(x-Tx,a)\right]+\frac{1}{8}\left[d(x-Tx,a)\right]^3}{\frac{1}{4}\left[d(x-Tx,a)\right]^2}$$

$$+\gamma\left[\frac{d(x-Tx,a)+d(y-Ty,a)}{2}\right]+\delta\left[\frac{\frac{1}{2}d(x-Tx,a)+\frac{1}{2}d(x-Tx,a)}{2}\right]$$

$$+\frac{\eta}{2}d(x-Tx,a)$$

$$\geq 2\alpha d(y-Ty,a)+\beta d(y-Ty,a)+\frac{\beta}{2}d(x-Tx,a)$$

$$+\gamma\left[\frac{d(x-Tx,a)+d(y-Ty,a)}{2}\right]+\frac{\delta}{2}d(x-Tx,a)+\frac{\eta}{2}d(x-Tx,a)$$

$$\geq d(x-Tx,a)\left(\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}\right)+d(y-Ty,a)\left(2\alpha+\beta+\frac{\gamma}{2}\right)$$

$$\geq \frac{1}{2}d(x-Tx,a)(\beta+\gamma+\delta+\eta)+\frac{1}{2}d(y-Ty,a)(4\alpha+2\beta+\gamma)$$

Now

$$d(z-u,a)=d(z-x,a)+d(x-u,a)$$

$$\geq \frac{1}{2}d(x-Tx,a)(3\beta+\gamma+\delta+\eta)+\frac{1}{2}d(y-Ty,a)(4\alpha+4\beta+\gamma+\delta)$$

$$+\frac{1}{2}d(x-Tx,a)(\beta+\gamma+\delta+\eta)+\frac{1}{2}d(y-Ty,a)(4\alpha+2\beta+\gamma)$$

$$\geq \frac{1}{2}d\left(x-Tx,a\right)\left(3\beta+\gamma+\delta+\eta+\beta+\gamma+\delta+\eta\right)$$
$$+\frac{1}{2}d\left(y-Ty,a\right)\left(4\alpha+4\beta+\gamma+\delta+4\alpha+2\beta+\gamma\right)$$

$$\geq \frac{1}{2}d\left(x-Tx,a\right)\left(4\beta+2\gamma+2\delta+2\eta\right)$$
$$+\frac{1}{2}d\left(y-Ty,a\right)\left(8\alpha+6\beta+2\gamma+\delta\right)-----------------(1.3)$$
$$d\left(z-u,a\right)=d\left(T\left(y\right)-T\left(2y-z\right),a\right)$$
$$=d(T(y)-2y+T(y),a)$$
$$=2d\left(Ty-y,a\right)--------------------(1.4)$$
So,

$$2d\left(Ty-y,a\right)\geq \frac{1}{2}d\left(x-Tx,a\right)\left(4\beta+2\gamma+2\delta+2\eta\right)+\frac{1}{2}d\left(y-Ty,a\right)\left(8\alpha+6\beta+2\gamma+\eta\right)$$

$$\Rightarrow\left[4-\left(8\alpha+6\beta+2\gamma+\eta\right)\right]d\left(y-Ty,a\right)\geq\left(4\beta+2\alpha+2\delta+2\eta\right)d\left(x-Tx,a\right)$$

$$\Rightarrow d\left(x-Tx,a\right)\leq\frac{4-\left(8\alpha+6\beta+2\gamma+\eta\right)}{4\beta+2\gamma+2\delta+2\eta}d\left(y-Ty,a\right)$$

$$\Rightarrow d\left(x-Tx,a\right)\leq kd\left(y-Ty,a\right)as\left(8\alpha+10\beta+4\gamma+2\delta+3\eta>4\right)$$

Where, $k=\dfrac{4-\left(8\alpha+6\beta+2\gamma+\eta\right)}{4\beta+2\gamma+2\delta+2\eta}<1$

Let $R=\frac{1}{2}\left(T+I\right)$, then

$$d\left(R^2\left(x\right)-R\left(x\right),a\right)=d\left(R\left(R\left(x\right)\right)-R\left(x\right),a\right)$$

$$=d\left(R\left(y\right)-y,a\right)=\frac{1}{2}d\left(y-Ty,a\right)$$
$$<\frac{k}{2}d\left(x-Tx,a\right)$$

By the definition of R we claim that $\left\{R^n\left(x\right)\right\}$ is a Cauchy sequence in X, $\left\{R^n\left(x\right)\right\}$ is converges to so element x_0 in X.

So $\lim_{n\to\infty}\left\{R^n\left(x\right)\right\}=x_0$. So $\left\{R\left(x_0\right)\right\}=x_0$

Hence $T\left(x_0\right)=x_0$

So x_0 is a fixed point of T.

Uniqueness

If possible let $y_0 \neq x_0$ is another fixed point of T. Then

$$d(x_0 - y_0, a) = d(Tx_0 - Ty_0, a)$$

$$\geq \alpha \frac{d(x_0 - Tx_0, a) d(y_0 - Ty_0, a)}{d(x_0 - y_0, a)}$$

$$+\beta \frac{d(y_0 - Ty_0, a) d(y_0 - Tx_0, a) d(x_0 - Ty_0, a) + [d(x_0 - y_0, a)]^3}{[d(x_0 - y_0, a)]^2}$$

$$+\gamma \left[\frac{d(x_0 - Tx_0, a) d(y_0 - Ty_0, a)}{2}\right]$$

$$+\delta \left[\frac{d(x_0 - Ty_0, a) + d(y_0 - Tx_0, a)}{2}\right]$$

$$+\eta d(x_0 - y_0, a)$$

$$\geq \beta d(x_0 - y_0, a) + \delta d(x_0 - y_0, a) + \eta d(x_0 - y_0, a)$$

$$\geq (\beta + \delta + \eta) d(x_0 - y_0, a)$$

Which is contradiction so $x_0 = y_0$

Hence fixed point is unique **Hence proved**

Theorem 3.2 :-

Let T and G be two non expansive mappings of a 2-metric space X into itself . T and G satisfy the conditions:

(2.1) T and G commute .

(2.2) $T^2 = I$ and $G^2 = I$, where I is identity mapping .

(2.3)

$$d(Tx - Ty, a) \geq \alpha \frac{d(Gx - Tx, \mathrm{a}) d(Gy - Ty, a)}{d(Gx - Gy, a)}$$

$$+\beta \frac{d(Gy - Ty, a) d(Gy - Tx, a) d(Gx - Ty, a) + [d(Gx - Gy, a)]^3}{[d(Gx - Gy, a)]^2}$$

$$+\gamma \left[\frac{d(Gx - Tx, a) + d(Gy - Ty, a)}{2}\right] + \delta \left[\frac{d(Gx - Ty, a) + d(Gy - Tx, a)}{2}\right] + \eta [d(Gx - Gy, a)]$$

For every $x, y \in X$, $\alpha, \beta, \gamma, \delta, \eta \in [0,1]$ with $x \neq y$ and $d(Gx, Gy) \neq 0$ and $\beta + \delta + \eta > 1$

Then there exist a Unique Common Fixed Point of T and G such that $T(x_0) = x_0$ and $G(x_0) = x_0$.

Proof :-

Suppose x is point in 2-metric space X it is clear that $(TG)^2 = I$

$$d(TG.G(x) - TG.G(y), a) \geq \alpha \frac{d(G(G^2x) - T(G^2x), a) d(G(G^2y) - T(G^2y), a)}{d(G(G^2x) - T(G^2y), a)}$$

$$+\beta \frac{d(G(G^2y) - T(G^2y), a) d(G(G^2y) - T(G^2x), a) d(G(G^2x) - T(G^2y), a) + [d(G(G^2x) - G(G^2y), a)]^3}{[d(G(G^2x) - G(G^2y), a)]^2}$$

$$+\gamma \left[\frac{d(G(G^2x) - T(G^2x), a) + d(G(G^2y) - T(G^2y), a)}{2} \right]$$

$$+\delta \left[\frac{d(G(G^2x) - T(G^2y), a) + d(G(G^2y) - T(G^2x), a)}{2} \right]$$

$$+\eta \left[d(G(G^2x) - G(G^2y), a) \right]$$

$$\geq \alpha \frac{d(Gx - TG(Gx), a) d(Gy - TG(Gy), a)}{d(Gx - Gy, a)}$$

$$+\beta \frac{d(Gy - TG(Gy), a) d(Gy - TG(Gx), a) d(Gx - TG(Gy)) + [d(Gx - Gy, a)]^3}{[d(Gx - Gy, a)]^2}$$

$$+\gamma \left[\frac{d(Gx - TG(Gx), a) + d(Gy - TG(Gy), a)}{2} \right]$$

$$+\delta \left[\frac{d(Gx - TG(Gx), a) + d(G(y) - TG(Gx), a)}{2} \right]$$

$$+\eta d(Gx - Gy, a)$$

Taking $G(x) = p, G(y) = q$, where $p \neq q$

$$\geq \alpha \frac{d(p-TG(p),a)d(q-TG(q),a)}{d(p-q,a)}$$
$$+\beta \frac{d(q-TG(q),a)d(q-TG(p),a)d(p-TG(q),a)+[d(p-q,a)]^3}{[d(p-q,a)]^2}$$
$$+\gamma\left[\frac{d(p-TG(p),a)+d(q-TG(q),a)}{2}\right]$$
$$+\delta\left[\frac{d(p-TG(q),a)+d(q-TG(p),a)}{2}\right]$$
$$+\eta d(p-q,a)$$

Taking $TG=R$ we get

$$d(R(p)-R(q),a)\geq \alpha \frac{d(p-R(p),a)d(q-R(q),a)}{d(p-q,a)}$$
$$+\beta \frac{d(q-R(q),a)d(q-R(p),a)d(p-R(q),a)+[d(p-q,a)]^3}{[d(p-q,a)]^2}$$
$$+\gamma\left[\frac{d(p-R(p),a)+d(q-R(p),a)}{2}\right]$$
$$+\delta\left[\frac{d(p-R(q),a)+d(q-R(p),a)}{2}\right]$$
$$+\eta d(p-q,a)$$

It is clear by theorem (3.1) ; that $R=TG$ has at least one fixed point say x_0 in K that is

$$R(x_0)=TG(x_0)=x_0$$

and so $T.(TG)(x_0)=T(x_0)$

$$G(x_0)=T(x_0)$$

Now

$$d(Tx_0-x_0,a)=d(Tx_0-T^2(x_0),a)=d(Tx_0-T.T(x_0),a)$$
$$\geq \alpha \frac{d(G(x_0)-T(x_0),a)d(GT(x_0)-T(Tx_0),a)}{d(G(x_0)-G(Tx_0),a)}$$
$$+\beta \frac{d(G(Tx_0)-T(Tx_0),a)d(G(Tx_0)-T(x_0),a)d(G(x_0)-T(Tx_0),a)+[d(G(x_0)-G(Tx_0),a)]^3}{[d(G(x_0)-G(Tx_0),a)]^2}$$
$$+\gamma\left[\frac{d(G(x_0)-T(x_0),a)+d(G(Tx_0)-T(Tx_0),a)}{2}\right]$$
$$+\delta\left[\frac{d(G(x_0)-T(Tx_0),a)+d(G(Tx_0)-T(x_0),a)}{2}\right]+\eta d(G(x_0)-G(Tx_0),a)$$
$$=(\beta+\delta+\eta)d(Tx_0-x_0,a)$$

So $T(x_0)=x_0 \qquad (\beta+\gamma+\eta>1)$

That is x_0 is the fixed point of T.

But $T(x_0)=G(x_0)$ so $G(x_0)=x_0$.

Hence x_0 is the fixed point of T and G .

Uniqueness :-

If possible let $y_0 \neq x_0$ is another common fixed point of T and G.

Then

$$d(x_0-y_0,a)=d\left(T^2(x_0)-T^2(y_0),a\right)=d\left(T\left(T(x_0)\right)-T\left(T(y_0)\right),a\right)$$

$$\geq \alpha \frac{d\left(G(Tx_0)-T(Tx_0),a\right)d\left(G(Ty_0)-T(Ty_0),a\right)}{d\left(G(Tx_0)-G(Ty_0),a\right)}$$

$$+\beta \frac{d\left(G(Ty_0)-T(Ty_0),a\right)d\left(G(Ty_0)-T(Tx_0),a\right)d\left(G(Tx_0)-T(Ty_0),a\right)+\left[d\left(G(Tx_0)-G(Ty_0),a\right)\right]^3}{\left[d\left(G(Tx_0)-G(Ty_0),a\right)\right]^2}$$

$$+\gamma\left[\frac{d\left(G(Tx_0)-T(Tx_0),a\right)+d\left(G(Ty_0)-T(Ty_0),a\right)}{2}\right]$$

$$+\delta\left[\frac{d\left(G(Tx_0)-T(Ty_0),a\right)+d\left(G(Ty_0)-T(Tx_0),a\right)}{2}\right]+\eta d\left(G(Tx_0)-G(Ty_0),a\right)$$

$$\geq \beta d(x_0-y_0,a)+\delta d(x_0-y_0,a)+\eta d(x_0-y_0,a)$$

$$\geq (\beta+\delta+\gamma)d(x_0-y_0,a)$$

But $\beta+\delta+\eta>1$

So $x_0=y_0$.

So Common Fixed point is Unique.

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