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Numerical Comparison of Variational Iterative Method and a new modified Iterative Decomposition Method of solvingIntegro-differential Equations

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ABSTRACT : *In this paper, a modified iterative decomposition method is proposed to solve the nth order linear and nonlinear integro differential equations. The solution was obtained by decomposition of the solution of the integro differential equations and the initial approximation was obtained by the evaluation of the source term. Subsequent approximations were obtained by applying the nonlinear operator on the sum of previous solutions obtained. The results obtained confirmed the accuracy and efficiency of the method when compared with the other methods found in literature. Some examples were given to illustrate the variational iterative method and the modified iterative decomposition method.*

Keywords: *Integro-differential Equations (IDEs), Iterative Decomposition Method (IDM), Variational Iterative Method (VIM),*

I. INTRODUCTION

Integral and integro-differential equations arise in many scientific and engineering applications. A type of the equations can be obtained from converting initial value problems with prescribed initial values. However another type can be derived from boundary value problems with the given boundary conditions. It is important to point out that converting initial value problems to an integral equation and converting an integral equation to initial value problems are commonly used in the literature. However converting boundary value problem to an integral equation and integral equation to equivalent boundary value problems are rarely used. In recent time researchers have worked on integral equations and lots of discoveries have been recorded. Adomian G. (1994)[1] presented the decomposition method, Aslam, Noor (2008)[2] had the Solution of Integro Differential Equations by Variational Iterative Method, Daftardar-Gejji V. and Jafari H.(2006)[4] produced an iterative method for solving nonlinear functional equations, He J. H. (1999)[7] produced the Homotopy perturbation technique of solution, Hemeda A. A. (2012)[8] presentented a new Iterative Method of aplication to nth-Order Integro-Differential Equations.

II. TECHNIQUES AND METHODOLOGY

A. Standard Variational Integration Techniques

Consider the differential equation

$$Lu + Nu = g(x) \tag{1}$$

Where L and N are linear and non-linear operators respectively and g(x) is the source term and equation (1) is formed non-homogeneous.

According to Variational Iteration Method by Aslam Noor (2008)[2] we constructed a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \lambda \int_0^x (Lu_n(t) + Nu_n(t) - g(t))dt \qquad (2)$$

Where λ is a general Lagrange multiplier which is identified optimally via variational theory, noting that in this method λ may be a constant or a function and $U_n(x)$ is a restricted value. The subscript n denotes the nth approximation. Hence, $\delta u_n = 0$ where δ is a variational derivative. For a complete use of the VIM, we followed two steps

1. The determination of the Lagrange multiplier λ that is identified optimally and
2. With λ determined, we substituted the result in (2) where the restriction is omitted.

Taking the variation with respect to independent variable u_n, we have.

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}(\int_0^x \lambda(t)(Lu_n(t) + Nu_n(t) - g(t))dt) \qquad (3)$$

or equivalently

$$\delta U_{n+1} = \delta u_n + \delta(\int_0^x \lambda(t)(Lu_n(t)dt) \qquad (4)$$

Integration by parts is usually used for the determination of $\lambda(t)$. In other word, we used

$$\int_0^x \lambda(t)u_n'(t)dt = \lambda(t)u_n(t) - \int_0^x \lambda'(t)u_n(t)dt \qquad (5)$$

$$\int_0^x \lambda(t)u_n''(t)dt = \lambda(t)u_n'(t) - \lambda'(t)u_n(t) + \int_0^x \lambda''(t)u_n(t)dt \qquad (6)$$

$$\int_0^x \lambda(t)u_n'''(t)dt = \lambda(t)u_n''(t) - \lambda'(t)u_n'(t) + \lambda''(t)u_n(t) - \int_0^x \lambda'''(t)u_n(t)dt \qquad (7)$$

and so on. These identities are obtained by integration by parts. E.g If $Lu_n(t) = u_n'(t)$ in (3) then (4) becomes

$$\delta u_{n+1} = \delta u_n + \delta\left\{\int_0^x \lambda(t)(Lu_n(t)dt)\right\} \qquad (8)$$

or

$$\delta u_{n+1} = \delta u_n(t)(1+\lambda)|_{t-x} - \int_0^x \lambda'(t)\delta u_n(t)dt \qquad (9)$$

but the extremum condition requires that δu_{n+1} on the LHS of (8) is zero and as a result, the RHS should be zero as well and this yields the stationary condition. This gives:

$$\lambda = -1 \qquad (10)$$

Also if $Lu_n(t) = u_n''(t)$ in (8), then it becomes

$$\delta u_{n+1} = \delta u_n + \delta(\int_0^x \lambda(t)(Lu_n(t)dt)) \qquad (11)$$

Integrating the integral of (11) by parts, we have:

$$\delta u_{n+1} = \delta u_n + \delta\lambda(u_n)'|_0^x - (\lambda'\delta u_n)|_o^x + \int_0^x \lambda''\delta u_n dt \qquad (12)$$

or

$$\delta u_{n+1} = \delta u_n(t)(1-\lambda'|_{t=x} + \delta\lambda(u_n)'_{t=x} + \int_0^x \lambda''\delta u_n dt \qquad (13)$$

where $\delta u_{n+1} = 0$, and these yield stationary condition

$$1-\lambda'|_{t=x} = 0, \lambda(t)|_{t=x}, \lambda''|_{t=x} = 0 \qquad (14)$$

$$\lambda = t - x \tag{15}$$

The Langrange multiplier is determined from the the general formula

$$\lambda(t) = \frac{(-1)^i (t-x)^{i-1}}{(i-1)!} \tag{16}$$

Where i is the order of the differential equation. Having determined the Langrange multiplier $\lambda(t)$, the successive approximation $u_{n+1}(x), n \geq 0$ of the solution $u(x)$ is readily obtained using selective function $u_0(x)$, However for fast convergence the function $u_0(x)$ should be selected by using the initial conditions as follows.

$$\left.\begin{array}{c} u_0(x) = u(0), \text{ for the first order,} \\ u_0(x) = u(0) + xu'(0), \text{ for the second order.} \\ u_0(x) = u(0) + xu'(0) + \dfrac{x^2}{2}u''(0), \text{ for the third order} \\ . \\ . \\ . \\ u_0(x) = u(0) + xu'(0) + \dfrac{x^2}{2}u''(0) + \ldots + \dfrac{x^{n-1}}{n-1}u^{n-1}(0), \text{ for the nth order} \end{array}\right\} \tag{17}$$

Consequently the solution

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{18}$$

In other words, the correct functional (2) will give several approximate solution which will tend toward the exact solution. The correct functional for the IDE is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(u_n^{(n)}(t) - f(t) - \int_0^t K(x,r)u_n(r)dr)dt \tag{19}$$

If $n = 0,1,2,\cdots k-1$ then equation (19) becomes

$$\left.\begin{array}{c} u_1(x) = u_0(x) + \int_0^x \lambda\left(u_0^{(n)}(t) - f(t) - \int_0^t K(t,r)u_0(r)dr\right)dt \\ u_2(x) = u_1(x) + \int_0^x \lambda\left(u_1^{(n)}(t) - f(t) - \int_0^t K(t,r)u_1(r)dr\right)dt \\ . \\ . \\ . \\ u_k(x) = u_{k-1}(x) + \int_0^x \lambda\left(u_{k-1}^{(n)}(t) - f(t) - \int_0^t K(x,r)u_{k-1}(r)dr\right)dt \end{array}\right\} \tag{20}$$

B. Derivation of the Modified IDM on General Problem

The general nth-order integro-differential equation is

$$y^{(n)} + f(x)y(x) + \int_{h(x)}^{g(x)} k(x,t)y^q y^{(m)}dt = g(x) \tag{21}$$

with

$$y(a)=\alpha_0, y'(a)=\alpha_1, y''(a)=\alpha_2,..., y^{(n-1)}(a)=\alpha_{(n-1)} \quad (22)$$

where $\alpha_i, i=0,1,...i-1$ are real constants. m, n and q are integers with $q \le m < n$ in (21). The function f(x), g(x) and k(x,t) are given and the unknown function $y(x)$ to be determined, we assumed that equation (21) has a unique solution.

Let us consider the following general non-linear equation

$$y = f + N(y) \quad (23)$$

where N is a non linear operator from a Banach space $B \to B$ and f is a known function, we assumed an approximate solution y of equation (23) of the form

$$y = \sum_{n=0}^{\infty} y_n \quad (24)$$

The non-linear operator N is defined as

$$N(\sum_{n=0}^{\infty} y_n) = N(y_0) + \sum_{n=1}^{\infty}\left\{N(\sum_{k=0}^{\infty} y_k) - N(\sum_{k=0}^{n-1} y_k)\right\} \quad (25)$$

Thus, from equations (24) and (25), equation (23) is equivalent to

$$\sum_{n=0}^{\infty} y_n = f + N(y_0) + \sum_{n=1}^{\infty}\left\{N(\sum_{k=0}^{\infty} y_k) - N(\sum_{k=0}^{n-1} y_k)\right\} \quad (26)$$

We defined the recurrence relation as

$$\left.\begin{aligned} y_0 &= f \\ y_1 &= N(y_0) \\ y_2 &= N(y_0+y_1) - N(y_0) \\ y_3 &= N(y_0+y_1+y_2) - N(y_0+y_1) \\ &\vdots \\ y_{n+1} &= N(y_0+y_1+y_2+...+y_n) - N(y_0+y_1+...+y_{n-1}) \end{aligned}\right\} \quad (27)$$

The nth-term approximate solution for equation (27) is given as:

$$y = y_0 + y_1 + y_2 + ... + y_{n-1} \quad (28)$$

III. NUMERICAL EXAMPLES

In this section, we demonstrated the standard VIM and the Modified IDM on some numerical examples. We considered the comparison of the methods in four examples in which two are linear and the other two are non-linear integro-differential equations and the degree of approximation; n = 6. The errors were also considered.

Example 1

Consider

$$u'(x) = 1 - \frac{x}{3} + \int_0^1 xtu(t)dt, \quad u(0)=0$$

The Exact solution is given as: $u(x) = x$

Table 1: Numerical Results of VIM and Modified IDM for Example 1 and the Errors obtained

x	Exact	VIM Result	IDM Result	Error for VIM	Error for IDM
0.0	0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.1	0.0999999936	0.09999999921	6.36 E-09	7.90 E-10
0.2	0.2	0.1999999746	0.1999999968	2.45 E-08	3.20 E-09
0.3	0.3	0.2999999428	0.2999999928	5.72 E-08	7.20 E-09
0.4	0.4	0.3999998983	0.3999999873	1.02 E-07	1.27 E-08
0.5	0.5	0.4999998411	0.4999999801	1.59 E-07	1.99 E-08
0.6	0.6	0.5999997711	0.5999999714	2.29 E-07	2.86 E-08
0.7	0.7	0.6999996885	0.6999999611	3.12 E-07	3.89 E-08
0.8	0.8	0.7999995931	0.7999999491	4.07 E-07	5.09 E-08
0.9	0.9	0.8999994850	0.8999999356	5.15 E-07	6.44 E-08
1.0	1.0	0.9999993420	0.9999999205	6.58 E-07	7.95 E-08

Example 2

Consider the second order integro differential equation

$$y''(x) = 2 - \frac{x}{2} + \int_0^1 xy(t)y'(t)dt$$

with initial conditions $y(0) = y'(0) = 0$; the Exact solution is given as $y(x) = x^2$

Table 2: Numerical Results of VIM and Modified IDM for Example 2 and the Errors obtained

x	Exact	VIM Result	IDM Result	Error for VIM	Error for IDM
.0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.0100000000	0.0099938228	0.0099999980	6.18 E-06	1.70 E-09
0.2	0.0400000000	0.0399505826	0.0399999860	4.94 E-05	1.36 E-08
0.3	0.0900000000	0.0898332164	0.0899999540	1.67 E-04	4.58 E-08
0.4	0.1600000000	0.1596046611	0.1599998910	3.95 E-04	1.09 E-07
0.5	0.2500000000	0.2492278537	0.2499997880	7.72 E-04	2.12 E-07
0.6	0.3600000000	0.3586657312	0.3599996330	1.33 E-03	3.67 E-07
0.7	0.4900000000	0.4878812305	0.4899994180	2.12 E-03	5.82 E-07
0.8	0.6400000000	0.6368372887	0.6399991310	3.16 E-03	8.69 E-07
0.9	0.8100000000	0.8054968427	0.8099987620	4.50 E-03	1.24 E-06
1.0	1.0000000000	0.9938228295	0.9999983020	6.18 E-03	1.70 E-06

Example 3

Consider the third order volterra fredhom integro differential equation, giving as

$$u'''(x) = 2sinx - x - 3\int_0^x (x-t)u(t)dt + \int_0^1 u(t)dt$$

with initial conditions

$$u(0) = u'(0) = 1, u''(0) = -1$$

the exact solution is giving as

$$u(x) = sinx + cosx$$

Table 3: Numerical Results of VIM and Modified IDM for Example 3 and the Errors obtained

x	Exact	VIM Result	IDM Result	Error for VIM	Error for IDM
0.0	1.0000000000	1.0000000000	1.000000000	0.000000000	0.000000000
0.1	1.0948375820	1.0951049000	1.095105400	2.67318 E-04	2.67318 E-04
0.2	1.1787359090	1.1808661320	1.180870200	2.13022 E-03	2.13022 E-03
0.3	1.2508566960	1.2579985000	1.258012100	7.14180 E-03	7.14180 E-03
0.4	1.3104793360	1.3272464300	1.327278500	1.67671 E-02	1.67671 E-02
0.5	1.3570081000	1.3893426000	1.389405200	3.23345 E-02	3.23345 E-02
0.6	1.3899780880	1.4449622900	1.445069300	5.49842 E-02	5.49842 E-02
0.7	1.4090598740	1.4946734600	1.494841000	8.56136 E-02	8.56136 E-02
0.8	1.4140628000	1.5388851500	1.539128000	1.24822 E-01	1.24822 E-01
0.9	1.4049368780	1.5777897200	1.578119300	1.72853 E-01	1.72853 E-01
1.0	1.3817732910	1.6113047500	1.611722800	2.29531 E-01	2.29531 E-01

Example 4

$$u^{iv}(x) = 1 + \int_0^x e^{-x} u^2(t)\,dt$$

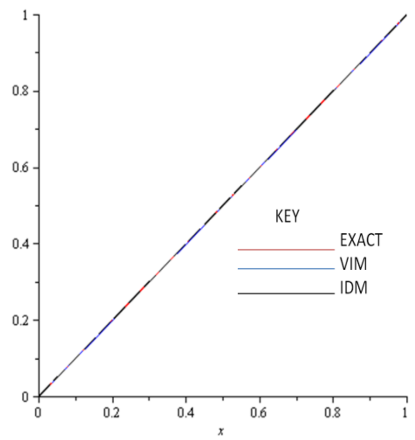
$$u(0) = u'(0) = 1, u(1) = e, u''(1) = e \quad Exact : u(x) = e^x$$

Table 4: Numerical Results of VIM and Modified IDM for Example 4 and the Errors obtained

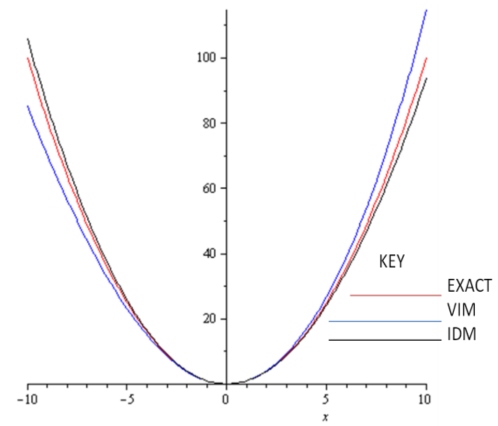
x	Exact	VIM Result	IDM Result	Error for VIM	Error for IDM
0.0	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000
0.1	1.1051581800	1.1161649000	1.1051707990	1.1000 E-02	1.1908 E-07
0.2	1.2214022700	1.2323962000	1.2214026690	1.0994 E-02	8.9160 E-08
0.3	1.3498588000	1.3608493000	1.3498580830	1.0991 E-02	7.4246 E-07
0.4	1.4918246900	1.5028023000	1.4918184720	1.0977 E-02	6.2256 E-06
0.5	1.6487212700	1.6596591000	1.6486976610	1.0938 E-02	2.3610 E-05
0.6	1.8221188000	1.8329540000	1.8220479640	1.0835 E-02	7.0836 E-05
0.7	2.0137527000	2.0243571000	2.0135713730	1.0604 E-02	1.8133 E-04
0.8	2.2255409200	2.2356765000	2.2251306430	1.0135 E-02	4.1029 E-04
0.9	2.4596031100	2.4688607000	2.4587584240	9.2576 E-03	8.4469 E-04
1.0	2.7182818200	2.7259981000	2.7166666020	7.7163 E-03	1.6158 E-03

IV. GRAPHS OF COMPARISON

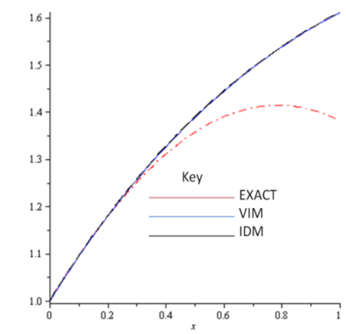
GRAHICAL SOLUTION FOR PROBLEM 1



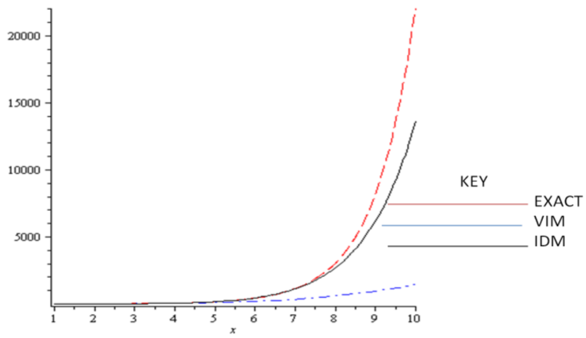
GRAPHICAL PROBLEM FOR PROBLEM 2



GRAPHICAL SOLUTION FOR EXAMPLE 3



GRAPHICAL SOLUTION FOR PROBLEM 4



V. DISCUSSION OF RESULTS AND CONCLUSION

It is very clear and easy to observe that the newly proposed method performed creditably well when it was applied to solve integro-differential equation. It compete favourably with the existing variational iteration method in the examples considered, we discovered that the results obtained using the modified iterative decomposition method are in close agreement with the exact solution and performed better than the other existing methods. in the case of Fredholm integro differential equations, when the newly proposed scheme was applied, we saw a great convergence to the exact solutions, infact there is a better performance in terms of errors obtained and closeness to the exact solutions than the variational iteration method. The modified iterative decomposition method performed creditably well when applied to solve integro-differential equations, infact it had an edge over the VIM. Moreover the new modified iterative decomposition method gives better results than those obtained by the variational iteration method, evident are shown in the table of results

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