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Location of Zeros of Analytic Functions

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Abstract: *In this paper we are finding the number of zeros of class of analytic functions, by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros of those of the existing results.*
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I. INTRODUCTION

Let P(z)= $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_n$ then all the zeros of P(z) lie in |z|≤1. Finding approximate zeros of a polynomial related to an analytic function is an important and well-studied problem. To find the number of zeros of a polynomial related to an analytic function has already been proved by Aziz and Mohamad [3], by extending Eneström-Kakeya theorem [1-2].

Here we establish the following results which are more interesting.

Theorem 1. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be an analytic function in* $|z| \le 1$ *such that*

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2,\ldots,n,\ldots \; for\ some\ real\ \beta, a_0 \ne 0,$$
$$0 < \tau \le 1, k \ge 0, k_1 \ge 0\ and$$
$$\tau|a_0| \le |a_1| \le \cdots \le |a_{m-2}| \le |a_{m-1}| \le k_1|a_m| \ge |a_{m+1}| \le |a_{m+2}| \ge |a_{m+3}| \le |a_{m+4}| \ge \cdots$$
$$\le |a_{n-2}| \ge |a_{n-1}| \le k|a_n| \ge |a_{n+1}| \ge |a_{n+2}| \ge \cdots$$

for some m, $0 \le m \le n$ then the number of zeros of F(z) in $|z| \le r$, $0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \frac{2\left[E_1 + |a_0| + \sin\alpha \sum_{i=1}^{\infty}|a_i| + (k_1|a_m| + k|a_n|)(1 + \cos\alpha + \sin\alpha) - \frac{1}{2}\tau|a_0|(1 + \cos\alpha - \sin\alpha)\right]}{|a_0|},$$

where $E_1 = [(|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|) - (|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|)]\cos\alpha - [\,|a_m| + |a_n|](1 + \sin\alpha)$.

Corollary 1. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ *be an analytic function in* $|z| \le 1$ *such that*

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2,\ldots,n,\ldots \; for\ some\ real\ \beta, a_0 \ne 0\ and$$
$$|a_0| \le |a_1| \le \cdots \le |a_{m-2}| \le |a_{m-1}| \le |a_m| \ge |a_{m+1}| \le |a_{m+2}| \ge |a_{m+3}| \le |a_{m+4}| \ge \cdots$$
$$\le |a_{n-2}| \ge |a_{n-1}| \le |a_n| \ge |a_{n+1}| \ge |a_{n+2}| \ge \cdots$$

for some m, $0 \le m \le n$ then the number of zeros of F(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{\left[E_2 + \sin\alpha \sum_{i=1}^{\infty}|a_i| + (|a_m| + |a_n|)\cos\alpha + \frac{1}{2}|a_0|(1 + \sin\alpha - \cos\alpha)\right]}{|a_0|},$$

where $E_2 = [(|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|) - (|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|)]\cos\alpha$.

Remark 1. By taking $\tau = k = k_1 = 1$ and $r = \frac{1}{2}$ in theorem 1, then it reduces to Corollary 1.

Theorem 2. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ $be\ an\ analytic\ function\ in$ $|z| \le 1$ $such\ that$

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2, \dots, n, \dots\ for\ some\ real\ \beta, a_0 \ne 0,$$
$$k \ge 0, 0 < \tau \le 1, 0 < \vartheta \le 1 and$$
$$k|a_0| \ge |a_1| \ge \cdots \ge |a_{m-2}| \ge |a_{m-1}| \ge \tau|a_m| \le |a_{m+1}| \ge |a_{m+2}| \le |a_{m+3}| \ge |a_{m+4}| \le \cdots$$
$$\ge |a_{n-2}| \le |a_{n-1}| \ge \vartheta|a_n| \le |a_{n+1}| \ge |a_{n+2}| \ge |a_{n+3}| \ge \cdots$$

for some m, $0 \le m \le n$ then the number of zeros of F(z) in $|z| \le r$, $0 < r < 1$
does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \frac{2\left[E_3 + \frac{1}{2}k|a_0|(1 + \cos\alpha + \sin\alpha) - (\tau|a_m| + \vartheta|a_n|)(1 + \cos\alpha - \sin\alpha) + \sin\alpha \sum_{i=1}^{\infty} |a_i|\right]}{|a_0|}.$$

$where\ E_3 = [(|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|) - (|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|)]\cos\alpha +$
$[\,|a_m| + |a_n|](1 - \sin\alpha).$

Corollary 2. Let $F(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ $be\ an\ analytic\ function\ in$ $|z| \le 1$ $such\ that$

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}, i = 0,1,2, \dots, n, \dots\ for\ some\ real\ \beta, a_0 \ne 0, and$$
$$|a_0| \ge |a_1| \ge \cdots \ge |a_{m-2}| \ge |a_{m-1}| \ge |a_m| \le |a_{m+1}| \ge |a_{m+2}| \le |a_{m+3}| \ge |a_{m+4}| \le \cdots$$
$$\ge |a_{n-2}| \le |a_{n-1}| \ge |a_n| \le |a_{n+1}| \ge |a_{n+2}| \ge |a_{n+3}| \ge \cdots$$

for some m, $0 \le m \le n$ then the number of zeros of F(z) in $|z| \le r$, $0 < r < 1$
does not exceed

$$\frac{1}{\log\frac{1}{r}} \log \frac{2\left[E_4 + \frac{1}{2}|a_0|(1 + \cos\alpha + \sin\alpha) - (|a_m| + |a_n|)\cos\alpha + \sin\alpha \sum_{i=1}^{\infty} |a_i|\right]}{|a_0|}.$$

$where\ E_4 = [(|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|) - (|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|)]\cos\alpha.$

Remark 2. By taking $k = \tau = \vartheta = 1$ in theorem 2, then it reduces to Corollary 2.

II. Lemmas

Lemma 1. [4]: Let $P(z) = \sum_{i=0}^{\infty} a_i z^i \not\equiv 0$ $be\ an\ analytic\ function\ in$ $|z| \le 1$ $such\ that$

$$|\arg(a_i) - \beta| \le \alpha \le \frac{\pi}{2}; |a_{i-1}| \le |a_i|\ \ for\ i = 0,1,2, \dots, n, \dots$$

$$\text{then } |a_i - a_{i-1}| \le (|a_i| - |a_{i-1}|)\cos\alpha + (|a_i| + |a_{i-1}|)\sin\alpha.$$

Lemma 2. [5]: If f(z) is regular $f(0) \ne 0$ and $f(z) \le M$ in $|z| \le 1$, $then\ the\ nuber\ of\ zeros\ of\ f(z)$ in $|z| \le r, 0 < r < 1$ $does\ not\ exceed$ $\frac{1}{\log\frac{1}{r}} \log \frac{M}{|a_0|}$.

III. Proofs of the Theorems

Proof of the Theorem 1.

Let $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + \cdots + a_n z^n + \cdots$ be an analytic function.

Let us consider the polynomial $G(z) = (1 - z)F(z)$ so that

$$G(z) = (1 - z)(a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + \cdots + a_n z^n + \cdots)$$
$$= a_0 + \sum_{i=1}^{\infty} (a_i - a_{i-1}) z^i$$

Now for $|z| \le 1$, we have

$$|G(z)| \le |a_0| + |a_0 - \tau a_0 + \tau a_0 - a_1| + \sum_{i=2}^{m-1} |\, a_i - a_{i-1}\,| + |\, a_m - k_1 a_m + k_1 a_m - a_{m-1}\,|$$
$$+ |\, a_{m+1} - k_1 a_m + k_1 a_m - a_m\,| + \sum_{i=m+2}^{n-1} |\, a_i - a_{i-1}\,| + |\, a_n - k a_n + k a_n - a_{n-1}\,|$$
$$+ |\, a_{n+1} - k a_n + k a_n - a_n\,| + \sum_{i=n+2}^{\infty} |\, a_i - a_{i-1}\,|$$
$$\le |a_0| + (1-\tau)|a_0| + |\tau a_0 - a_1| + \sum_{i=2}^{m-1} |\, a_i - a_{i-1}\,| + 2(k_1 - 1)|a_m| + |k_1 a_m - a_{m-1}\,| + |\, k_1 a_m - a_{m+1}\,|$$
$$+ \sum_{i=m+2}^{n-1} |\, a_i - a_{i-1}\,| + 2(k-1)|a_n| + |k a_n - a_{n-1}\,| + |\, k a_n - a_{n+1}\,| + \sum_{i=n+2}^{\infty} |\, a_i - a_{i-1}\,|$$

By using lemma 1 we get

$$|G(z)| \le |a_0| + (1-\tau)|a_0| + (\,|a_1| - \tau|a_0|\,)\cos\alpha + (\,|a_1| + \tau|a_0|\,)\sin\alpha + 2[\,(k_1 - 1)|a_m| + (k-1)|a_n|]$$
$$+ \sum_{i=2}^{m-1} (\,|\, a_i| - |a_{i-1}|\,)\cos\alpha + \sum_{i=2}^{m-1} (\,|\, a_i| + |a_{i-1}|\,)\sin\alpha + (\,k_1|a_m| - |a_{m-1}|\,)\cos\alpha$$
$$+ (\,k_1|a_m| + |a_{m-1}|\,)\sin\alpha + (\,k_1|a_m| - |a_{m+1}|\,)\cos\alpha + (\,k_1|a_m| + |a_{m+1}|\,)\sin\alpha$$
$$+ (|a_{m+2}| - |a_{m+1}|)\cos\alpha + (\,|a_{m+2}| + |a_{m+1}|\,)\sin\alpha + (\,|a_{m+2}| - |a_{m+3}|\,)\cos\alpha$$
$$+ (\,|a_{m+2}| + |a_{m+3}|\,)\sin\alpha + \cdots + (\,|a_{n-2}| - |a_{n-3}|\,)\cos\alpha + (\,|a_{n-2}| + |a_{n-3}|\,)\sin\alpha$$
$$+ (\,|a_{n-2}| - |a_{n-1}|\,)\cos\alpha + (\,|a_{n-2}| + |a_{n-1}|\,)\sin\alpha + (k\,|a_n| - |a_{n+1}|\,)\cos\alpha$$
$$+ (k\,|a_n| + |a_{n-1}|\,)\sin\alpha + (k\,|a_n| - |a_{n+1}|\,)\cos\alpha + (k\,|a_n| + |a_{n+1}|\,)\sin\alpha$$
$$+ \sum_{i=n+2}^{\infty} (\,|a_{i-1}| - |a_i|\,)\cos\alpha + \sum_{i=n+2}^{\infty} (\,|\, a_{i-1}| + |a_i|\,)\sin\alpha$$

$$= |a_0| + (1-\tau)|a_0| + (\,|a_1| - \tau|a_0|\,)\cos\alpha + (\,|a_1| + \tau|a_0|\,)\sin\alpha + 2[\,(k_1 - 1)|a_m| + (k-1)|a_n|]$$
$$+ (|a_{m-1}| - |a_1|)\cos\alpha + \sum_{i=2}^{m-1} (\,|\, a_i| + |a_{i-1}|\,)\sin\alpha + (\,k_1|a_m| - |a_{m-1}|\,)\cos\alpha$$
$$+ (\,k_1|a_m| + |a_{m-1}|\,)\sin\alpha + (\,k_1|a_m| - |a_{m+1}|\,)\cos\alpha + (\,k_1|a_m| + |a_{m+1}|\,)\sin\alpha$$
$$+ 2[(|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|) - (|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}|$$
$$+ |a_{n-1}|)]\cos\alpha + |\, a_{m+1}|\sin\alpha + \sum_{i=m+2}^{n-2} 2|\, a_i|\sin\alpha + |\, a_{n-1}|\sin\alpha + (k\,|a_n| - |a_{n-1}|\,)\cos\alpha$$
$$+ (k\,|a_n| + |a_{n-1}|\,)\sin\alpha + (k\,|a_n| - |a_{n+1}|\,)\cos\alpha + (k\,|a_n| + |a_{n+1}|\,)\sin\alpha + |a_{n+1}|\cos\alpha$$
$$+ \sum_{i=n+2}^{\infty} (\,|\, a_i| + |a_{i-1}|\,)\sin\alpha$$
$$= 2|a_0| - \tau|a_0|(1 + \cos\alpha - \sin\alpha) + 2(k_1|a_m| + k|a_n|)(1 + \cos\alpha + \sin\alpha) - 2[\,|a_m| + |a_n|](1 + \sin\alpha)$$
$$+ 2\sin\alpha \sum_{i=1}^{\infty} |\, a_i|$$
$$+ 2[(|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|) - (|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}|$$
$$+ |a_{n-1}|)]\cos\alpha$$

$$\le 2\left[E_1 + |a_0| + \sin\alpha \sum_{i=1}^{\infty} |\, a_i| + (k_1|a_m| + k|a_n|)(1 + \cos\alpha + \sin\alpha) - \frac{1}{2}\tau|a_0|(1 + \cos\alpha - \sin\alpha)\right],$$

where $E_1 = [(|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|) - (|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|)]\cos\alpha - [\,|a_m| + |a_n|](1 + \sin\alpha)$

Apply lemma 2 to G(z), we get then number of zeros of G(z) in $|z| \le r, 0 < r < 1$ does not exceed

$$\frac{1}{\log\frac{1}{r}}\log\frac{2\left[E_1+|a_0|+\sin\alpha\sum_{i=1}^{\infty}|a_i|+(k_1|a_m|+k|a_n|)(1+\cos\alpha+\sin\alpha)-\frac{1}{2}\tau|a_0|(1+\cos\alpha-\sin\alpha)\right]}{|a_0|},$$

$where\ E_1=[(|a_{m+2}|+|a_{m+4}|+\cdots+|a_{n-4}|+|a_{n-2}|)-(|a_{m+1}|+|a_{m+3}|+\cdots+|a_{n-3}|+|a_{n-1}|)]\cos\alpha-[\,|a_m|+|a_n|](1+\sin\alpha)$

All the number of zeros of F(z) in $|z|\le r, 0<r<1$ is also equal to the number of zeros of G(z) in $|z|\le r, 0<r<1$.

This completes the proof of theorem 1.

Proof of the Theorem 2.

Let $\mathrm{F(z)}=a_0+a_1z+a_2z^2+\cdots+a_mz^m+\cdots+a_nz^n+\cdots$ be an analytic function.

Let us consider the polynomial $\mathrm{G(z)}=(1-\mathrm{z})F(z)$ so that

$$\mathrm{G(z)}=(1-\mathrm{z})(a_0+a_1z+a_2z^2+\cdots+a_mz^m+\cdots+a_nz^n+\cdots)$$
$$=a_0+\sum_{i=1}^{\infty}(a_i-a_{i-1})z^i$$

Now for $|z|\le 1$, we have

$$|\mathrm{G(z)}|\le|a_0|+|a_0-ka_0+ka_0-a_1|+\sum_{i=2}^{m-1}|a_i-a_{i-1}|+|a_m-\tau a_m+\tau a_m-a_{m-1}|$$
$$+|a_{m+1}-\tau a_m+\tau a_m-a_m|+\sum_{i=m+2}^{n-1}|a_i-a_{i-1}|+|a_n-\vartheta a_n+\vartheta a_n-a_{n-1}|$$
$$+|a_{n+1}-\vartheta a_n+\vartheta a_n-a_n|+\sum_{i=n+2}^{\infty}|a_i-a_{i-1}|$$
$$\le|a_0|+(k-1)|a_0|+|ka_0-a_1|+\sum_{i=2}^{m-1}|a_i-a_{i-1}|+2(1-\tau)|a_m|+|\tau a_m-a_{m-1}|+|\tau a_m-a_{m+1}|$$
$$+\sum_{i=m+2}^{n-1}|a_i-a_{i-1}|+2(1-\vartheta)|a_n|+|\vartheta a_n-a_{n-1}|+|\vartheta a_n-a_{n+1}|+\sum_{i=n+2}^{\infty}|a_i-a_{i-1}|$$

By using lemma 1 we get

$$|\mathrm{G(z)}|\le|a_0|+(k-1)|a_0|+(k|a_0|-|a_1|)\cos\alpha+(k|a_0|+|a_1|)\sin\alpha+2[(1-\tau)|a_m|+(1-\vartheta)|a_n|]$$
$$+\sum_{i=2}^{m-1}(|a_{i-1}|-|a_i|)\cos\alpha+\sum_{i=2}^{m-1}(|a_{i-1}|+|a_i|)\sin\alpha+(|a_{m-1}|-\tau|a_m|)\cos\alpha$$
$$+(|a_{m-1}|+\tau|a_m|)\sin\alpha+(|a_{m+1}|-\tau|a_m|)\cos\alpha+(|a_{m-1}|+\tau|a_m|)\sin\alpha$$
$$+(|a_{m+1}|-|a_{m+2}|)\cos\alpha+(|a_{m+1}|+|a_{m+2}|)\sin\alpha+(|a_{m+3}|-|a_{m+1}|)\cos\alpha$$
$$+(|a_{m+3}|+|a_{m+2}|)\sin\alpha+\cdots+(|a_{n-3}|-|a_{n-2}|)\cos\alpha+(|a_{n-3}|+|a_{n-2}|)\sin\alpha$$
$$+(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha+(|a_{n-1}|-\vartheta|a_n|)\cos\alpha$$
$$+(|a_{n-1}|+\vartheta|a_n|)\sin\alpha+(|a_{n+1}|-\vartheta|a_n|)\cos\alpha+(|a_{n+1}|+\vartheta|a_n|)\sin\alpha$$
$$+\sum_{i=n+2}^{\infty}(|a_{i-1}|-|a_i|)\cos\alpha+\sum_{i=n+2}^{\infty}(|a_i|+|a_{i-1}|)\sin\alpha$$

$$= |a_0| + (k-1)|a_0| + (\,k|a_0|-|a_1|\,)cos\alpha + (\,k|a_0|+|a_1|\,)sin\alpha + 2[\,(1-\tau)|a_m| + (1-\vartheta)|a_n|]$$
$$+ (|a_1| - |a_{m-1}|)cos\alpha + \sum_{i=2}^{m-1} (\,|\,a_i| + |a_{i-1}|\,)\,sin\alpha + (\,|a_{m-1}|-\tau|a_m|\,)cos\alpha$$
$$+ (\,|a_{m-1}|+\tau|a_m|\,)sin\alpha + (\,|a_{m+1}|-\tau|a_m|\,)cos\alpha + (\,|a_{m-1}|+\tau|a_m|\,)sin\alpha$$
$$+ 2[(|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|)$$
$$- (|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|)]cos\alpha + |\,a_{m+1}|sin\alpha + \sum_{i=m+2}^{n-2} 2|\,a_i|\,sin\alpha$$
$$+ |\,a_{n-1}|\,sin\alpha + (\,|a_{n-1}|-\vartheta|a_n|\,)cos\alpha + (\,|a_{n-1}|+\vartheta|a_n|\,)sin\alpha + (\,|a_{n+1}|-\vartheta|a_n|\,)cos\alpha$$
$$+ (\,|a_{n+1}|+\vartheta|a_n|\,)sin\alpha + |a_{n+1}|cos\alpha + \sum_{i=n+2}^{\infty} (\,|\,a_i| + |a_{i-1}|\,)\,sin\alpha$$

$$= k|a_0|(1 + cos\alpha + sin\alpha) - 2(\tau|a_m| + \vartheta|a_n|)(1 + cos\alpha - sin\alpha) + 2[\,|a_m| + |a_n|](1 - sin\alpha)$$
$$+ 2[(|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|)$$
$$- (|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|)]cos\alpha + 2sin\alpha \sum_{i=1}^{\infty} |\,a_i|$$

$$\leq 2\left[E_3 + \frac{1}{2}k|a_0|(1 + cos\alpha + sin\alpha) - (\tau|a_m| + \vartheta|a_n|)(1 + cos\alpha - sin\alpha) + sin\alpha \sum_{i=1}^{\infty} |\,a_i|\right].$$

where $E_3 = [\,|a_m| + |a_n|](1 - sin\alpha)$
$+[(|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|) - (|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|)]cos\alpha.$

Apply lemma 2 to G(z), we get then number of zeros of G(z) in $|z| \leq r, 0 < r < 1$ does not exceed

$$\frac{1}{log\frac{1}{r}}log\frac{2\left[E_3 + \frac{1}{2}k|a_0|(1 + cos\alpha + sin\alpha) - (\tau|a_m| + \vartheta|a_n|)(1 + cos\alpha - sin\alpha) + sin\alpha \sum_{i=1}^{\infty} |\,a_i|\right]}{|a_0|}.$$

where $E_3 = [\,|a_m| + |a_n|](1 - sin\alpha)$
$+[(|a_{m+1}| + |a_{m+3}| + \cdots + |a_{n-3}| + |a_{n-1}|) - (|a_{m+2}| + |a_{m+4}| + \cdots + |a_{n-4}| + |a_{n-2}|)]cos\alpha.$

All the number of zeros of F(z) in $|z| \leq r, 0 < r < 1$ is also equal to the number of zeros of G(z) in $|z| \leq r, 0 < r < 1$

This completes the proof of theorem 2.

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