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Insight on the Projective Module

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Abstract : In this work, we focus on some important results on modules and projective modules. We discuss some aspects of modules laying emphasis on its axioms, exactness and free module.

Keywords: submodules, exactness, split exactness, homomorphism

1. Introduction

The theory of module is an important branch in Algebra. There are many applications of projective module as studied by some researchers. Yengui (2011) observed that for any finite-dimensional ring, all finitely generated stably free modules over $R[X]$ of rank $>$ dim R are free; the result was only known for Noetherian rings. According to Tütüncü et al. (2012) if M is a semi-projective w-coretractable module with finite hollow dimension and $S = \mathrm{End}R(M)$, there exist $h_1, \dots, h_n \in S$ such that $\mathrm{J}(S) = A_{h_1}, \dots, A_{h_n}$, where $\mathrm{Im}\ h_i$ is a nonzero hollow submodule of M_R and $A_{h_i} = \{s \in S | \mathrm{Im}\ s + \mathrm{Ker}\ h_i \neq M\}$ for each $1 \le i \le n$. The paper discusses concept of module, exactness of sequence. There are two classes of R-modules: left and right R-modules. The result on left R-modules yields a corresponding result on right R-modules. Since R is commutative, both left R-module and right R-module will be regarded as R-module.

Definition 1.1. Let R be a ring with identity. A left R-module is a set together with a binary operation "+" on M under which M is an abelian group, and an action on M, that is, a map $R \times M \to M : (r, x) \mapsto rm, \forall m \in M$, which satisfies the following axioms:

(a.) $(r + s)m = rm + sm$

(b.) $(rs)m = r(sm)$

(c.) $r(m+n) = rm + rn \; \forall \; r,s \in R$ and $m,n \in M$

(d.) $1 \cdot m = m \; \forall \; m \in M$.

Similarly, a right R-module M is an abelian group together with a map $M \times R \to M: (m,r) \mapsto mr, \forall m \in M$, satisfying the following axioms: For $r,s \in R, m,n \in M$

(a.) $m(r+s) = mr + ms$

(b.) $m(rs) = (mr)s$

(c.) $(m+n)r = mr + nr$

(d.) $m \cdot 1 = m \; \forall \; m \in M$.

Every abelian group is a $\mathbb{Z}$- module.

1.2. Submodule. Let M be an R-module. A non-empty subset N of M is called a Submodule of M if N is an additive subgroup of M closed under the "same" multiplication by elements of R, as for M. This implies that the axioms of an R-modules must be satisfied by N.

2. R-Homomorphisms between two modules

Let M and N be two R-modules, then a function $f: M \to N$ is a homomorphism in case $\forall a,b \in R$ and all $x,y \in M$; $f(ax+by) = af(x) + bf(y)$. The function $'f'$ must preserve the defining structure in order to be a module homomorphism. Precisely, if R is a ring and M,N are R-modules. A mapping $f: M \to N$ is called an R-module homomorphism if

(i) $f(x+y) = f(x) + f(y) \forall x,y \in M$.

(ii) $f(ax) = af(x) + bf(y)$.

2.1. Epimorphisms and Monomorphisms

A homomorphism $f: M \to N$ is called an epimorphism in case it is surjective. That is, if f maps the elements of module M onto N. An injective R-homomorphism $f: M \to N$ is called an R-monomorphism (that is, one-to-one). An R-homomorphism $f: M \to N$ is an R-isomorphism in case it is a bijection. Thus, two modules M and N are said to be isomorphic, denoted by $M \cong N$, if there is an R-isomorphism $f: M \to N$. An R-homomorphism $f: M \to M$ is called an R-endomorphism. If $f: M \to M$ is bijective, then, it is called an R-automorphism.

2.1.1 Kernel and Image of R-homomorphism. Let R be a ring. Let M and N be R-modules and $f: M \to N$ be an R- homomorphism. The Kernel of f denoted by $\text{Ker}\, f$ is given by $\text{Ker}(f) := \{x \in M | f(x) = 0_N\}$.

The image of f denoted by $\text{Im}(f)$ or $f(M)$ is defined as

$\text{Im}(f) := \{y \in N | y = f(x) \text{ for some } x \in M\}$.

Proposition 2.1.2.

Let M and N be R-modules and $f: M \to N$ be an R-module homomorphism.

(i) $\text{Ker}\, f$ is a submodule of M.

(ii) $\text{Im}\, f$ is a submodule of N.

(iii)The R-homomorphism $f: M \to N$ is a monomorphism if and only if $Ker\, f = 0_M$.

Proof: (i) Let $\alpha, \beta \in \text{Ker}\, f$. We claim that $\alpha - \beta \in \text{Ker}\, f$. Let 0_N be a zero of N. Then $f(\alpha) = 0_N$ and $f(\beta) = 0_N$ by the definition of Ker N. $\Rightarrow f(\alpha) = f(\beta)$ since 0_N is arbitrary. This implies that $f(\alpha) - f(\beta) = f(\alpha - \beta)$ since f is a homomorphism. $\Rightarrow \alpha - \beta \in \text{Ker}\, f$.

Again, let $r \in R$ and $k \in \text{Ker}\, f$. $f(k) = 0_N$. $\Rightarrow$ $r0_N = rf(K) = f(rK)$ since f is a homomorphism. Since M is a module, $rk \in \text{Ker}\, f$. Hence, $\text{Ker}\, f$ is a submodule of M.

(ii) Let $n_1, n_2 \in \text{Im}\, f$. Then, there exists $m_1, m_2 \in M$ such that $f(m_1) = n_1$, $f(m_2) = n_2$; $n_1 - n_2 = f(m_1) - f(m_2) = f(m_1 - m_2)$ since f is a homomorphism. But $m_1 - m_2 \in M$ (since M is a module). Therefore, $(n_1 - n_2) \in \text{Im}\, f$. Also, let $r \in R$ and $n \in \text{Im}\, f$. There exists $m \in M$ such that $f(m) = n$. $rn = rf(m) = f(rm)$, since f is a homomorphism. But $rm \in M$ since M is a module. Therefore, $rn = f(rm) \in \text{Im}\, f$.

Thus, $\text{Im}\, f$ is a submodule of N.

(iii) If: Let $Ker\, f = 0_M$ and $f(b) = f(c)$ for $b, c \in M$.

$f(b - c) = f(b) - f(c) = 0_N - 0_N = 0_N \Rightarrow b - c \in \text{Ker}\, f \Rightarrow b - c = 0 \Rightarrow b = c \Rightarrow$ f is a monomorphism.

Only if: Let f be a monomorphism and $x \in \text{Ker}\, f$. Then, $f(x) = 0_N = f(0_M)$ since f is a monomorphism and $x \in \text{Ker}\, f$, $x = 0_M$. □

2.2. Factor Module

Theorem and definition: Let M be an R-module. Let N be a submodule of M and $M/N = \{x + N | x \in M\}$. Then M/N is an R-module, where addition in M/N is defined by $(x + N) + (y + N) = (x + y) + N$ for any $x, y \in M$ and closure under multiplication by scalars in R is defined as follows: for $r \in R$; $x + N \in M/N$, $r(x + N) = rx + N$.

Proof: M/N is an additive abelian group. Also, closure by scalars in R follows from definition. Now, to verify module axioms. Generally, for $x \in M$, $\bar{x} = x + N$. Let $a, b \in R$.

(i) $(a + b)\bar{x} = (a + b)x + M = (ax + bx) + M = (ax + M) + (bx + M)$

$= a(x + M) + b(x + M) = a\bar{x} + b\bar{x}$.

(ii) $a(\bar{x} + \bar{y}) = a((x + N) + (y + N)) = a((x + y) + N) = (a(x + y)) + N$

$= (ax + ay) + N = (ax + N) + (ay + N) = a\bar{x} + a\bar{y}$.

(iii) $(ab)\bar{x} = ab(x + N) = ((ab)x) + N$ (by associativity in R)

$= a(bx + N) = a(b\bar{x})$.

If R has identity $'1'$, then, $1\bar{x} = 1(x + N) = (1x) + N = \bar{x}$. □

2.3 Exactness

2.3.1 Exact Sequence. Let $f: M' \to M$ and $g: M \to M''$. The pair of homomorphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be exact at M in case $\text{Im } f = \text{Ker } g$. That is, let $y \in M$, then $g(y) = 0$ if and only if there exists $x \in M''$ such that $f(x) = y$. In general, a finite or infinite sequence of homomorphism $\cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \cdots$ is exact in case it is exact for each successive pair f_n and f_{n+1}. $\Rightarrow$ $\text{Im } f_n = \text{Ker} f_{n+1}$. $\Rightarrow$ for any three consecutive terms, the subsequence $M_{n-1} \to M_n \to M_{n+1}$ is exact.

2.3.2 Short exact Sequence

An exact sequence of the form $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is called a short exact sequence.

Proposition 2.3.3. If the sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is a short exact sequence. Then, f is a monomorphism and g is an epimorphism. $\text{Ker } g = \text{Im } f$.

Proof. The exactness of $0 \to M \xrightarrow{f} N$ means that $\text{Ker } f = 0$. This implies that f is injective. Similarly, f is surjective is equivalent to $M \xrightarrow{f} N \to 0$ is exact.

Thus, in the given short exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$, $'f'$ is injective, $'g'$ is surjective and $\text{Ker } g = \text{Im } f$. $\Rightarrow f$ is a monomorphism and g is an epimorphism.

Remark. Let M and N be any two R-modules. Every R-homomorphism from M to N is an element of the set of functions from M to N. These homomorphisms form a set with standard notation; $\mathrm{HOM}_R(M,N)$.

Theorem 2.3.4. $\mathrm{HOM}_R(M,N)$ is an R-module, where R is a commutative ring and M,N are R-modules.

Proof: Suppose R is commutative. For $\alpha \in \mathrm{HOM}_R(M,N)$, $r \in R$, define $(r\alpha)$ by $(r\alpha)(x) = r(\alpha x)$. Since $r,s \in R$, $\alpha \in \mathrm{HOM}_R(M,N)$ and $x \in M$,

$$(r\alpha)(sx) = r(\alpha(sx)) = r\left(s(\alpha(x))\right) = rs(\alpha(x)) = r((s\alpha)(x)).$$

We have that $r\alpha \in \mathrm{HOM}_R(M,N)$. Also, let $f,g \in \mathrm{HOM}_R(M,N)$ and $f+g$ preserves the scalar multiplication. Let $\alpha \in R$ and $x \in M$ be arbitrarily given. Then, we have;

$$(f+g)(\alpha x) = f(\alpha x) + g(\alpha x) = \alpha f(x) + \alpha g(x) = \alpha[f(x) + g(x)] = [\alpha(f+g)(x)].$$

Hence, $f+g$ preserves scalar multiplication. Since R is commutative, for arbitrary given $\alpha \in R$ and $\mathrm{f} \in \mathrm{HOM}_R(M,N)$, we define a function $\alpha f: M \to N$ by taking $(\alpha f)(x) = \alpha[f(x)]$ for every $x \in M$. Thus, $\mathrm{HOM}_R(M,N)$ is an R-module. □

Remark. From the above theorem, if R is a commutative ring and M is an R-module, then, $\mathrm{HOM}_R(M,N)$ is also an R-module, where $N = R$ is considered an R-module over itself. $\mathrm{HOM}_R(M,N)$ is a module called the dual of M and it is denoted by M^*.

2.4 Direct Summands, Split Homomorphism

Suppose $M_1$and M_2 are submodules of a module M. M_1 and M_2 span M in case $M = M_1 + M_2$. M_1 and M_2 are linearly independent in case $M_1 \cap M_2 = 0$. Let $i: (x_1,x_2) \mapsto x_1 + x_2$, $((x_1,x_2) \in M_1 \times M_2)$ be a canonical R-homomorphism $'i'$ from the cartesian product $M_1 \times M_2$ module with $\mathrm{Im}\ i = M_1 + M_2$, $Ker\ i = \{(x,-x) | x \in M_1 \cap M_2\}$. Then, i is *epic* if and only if M_1 and M_2 span M, and *monic* if and only if M_1 and M_2 are independent. If $'i'$$ the canonical homomorphism is an isomorphism, then M is the (internal) direct sum of its submodules M_1 and M_2, and it is denoted by $M = M_1 \oplus M_2$. Thus, for each $m \in M$, there exists unique $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2 \in M$ if and only if $M = M_1 \oplus M_2$.

A submodule M_1 of M is a direct summand of M in case there is a submodule M_2 of M such that $M = M_1 \oplus M_2$. M_2 is also a direct summand. M_1 and M_2 are called *Complementary direct summands*. More generally, Let $M_1, \dots, M_n$ be submodules of an R-module M. Suppose $M = \sum_{i=1}^{n} M_i$ and each $x \in M$ has a unique representation $x = \sum_{i=1}^{n} x_i$, $x_i \in M_i$. Then, M is called the *internal direct sum* of M_i's and it is written as $M = \bigoplus_{i=1}^{n} M_i$.

2.4.1 Split exact Sequence

Let $f: M \to N$ and $g: N \to M$ be homomorphisms with $fg = 1_N$, then f is a split epi and we write $M \xrightarrow{f} N \to 0$ and we say g is a split mono, and write $0 \to N \xrightarrow{g} M$. Thus, a short exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is split or is split exact in case f is a split mono and g is a split epi.

Lemma 2.4.2. Let $f: M \to N$ and $f': N \to M$ be homomorphisms such that $ff' = 1_N$. Then, f is an epimorphism, f' is a monomorphism, and $M = Ker\ f \oplus Im\ f'$.

Proof: Clearly, f is epi, f' is monic. If $f'(y) = x \in \mathrm{Ker}\, f \cap \mathrm{Im}\, f'$ then $0 = f(x) = ff'(y) = y$, and $x = f'(y) = 0$. If $x \in M$, then $f(x - f'f(x)) = f(x) - f(x) = 0$ and

$x = x - f'f(x) + f'f(x) \in \mathrm{Ker} f + \mathrm{Im}\, f. \Rightarrow M = Ker\ f \oplus Im\ f'$. □

Proposition 2.4.3. The following statements about a short exact sequence

$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$

are equivalent in R^M.

(a.) The sequence is split;

(b.) The monomorphism $f: M_1 \to M$ is split;

(c.) The epimorphism $g: M \to M_2$ is split;

(d.) $\mathrm{Im}\ f = \mathrm{Ker}\ g$ is a direct summand of M;

(e.) Every homo $h: M_1 \to N$ factors through f;

(f.) Every homo $\bar{h}: N \to M_2$ factors through g.

3.1 Free Module

Definition 3.1.1.

Let S be a non-empty subset of an R-module M. S is called *a set of free generators* or *basis* or *base* for M if every element $x \in M$ can be uniquely expressed as a linear combination of elements of S. That is, $x = \sum a_y y$ uniquely where y ranges through the elements of S and only a finite number of a_y's are non-zero in R. A subset S of a module M over R is said to be *linearly independent* if and only if for every finite number of distinct elements $x_1, \dots, x_n \in S$, $\sum_{i=1}^{n} \alpha_i x_i = 0$, $\alpha_i \epsilon R$, $\Rightarrow \alpha_i = 0 \ \forall \ i = 1, \dots, n$. An R-module M is *free* if it has a basis. A non-zero cyclic module R_x is said to be *free* if given $a \in R, ax = 0 \ \Rightarrow a = 0$. Precisely, S is a basis for M if $M = \bigoplus_{x \in S} R_x$ where each R_x is a free cyclic module. Thus, S is a basis for M if and only if S is linearly independent and generates M.

Theorem 3.1.2. Given any R-module M, there exists an exact sequence

$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ such that F is a free R-module.

Proof: Let F be a free R-module with basis $S = M$. Define a mapping $f: S \to M$ by $f(x) = x$. Then, f extends to a unique homomorphism $g: F \to M$, g is surjective since $F = \langle M \rangle$. Let $K = \operatorname{Ker} g$. Then, $0 \longrightarrow K \xrightarrow{\alpha} F \xrightarrow{\varepsilon} M \longrightarrow 0$ is exact. □

Definition 3.1.3. An R-module M is said to be *finitely generated* (FG) if M can be generated by some finite set of elements. Thus, a module M is *finitely generated* if there is a finite subset N of M with $RN = M$. If M and M_1 are finitely generated, so is $M \oplus M_1$.

Theorem 3.1.4. Let S be a basis for a free R-module, F and N any R-module. Let $\alpha: S \to N$ be any mapping. Then, there exists a unique R-homomorphism, $\beta: F \to N$ such that $\beta|_S = \alpha$.

Proof: Let $z \in F$, then, z can be expressed in the form $z = \sum_{x \in S} a_x x$ where only a finite number of the a_x's are non-zero. Define a map $\beta: F \to N$ by $\beta(z) = \sum_{x \in S} a_x \alpha(x)$. Thus, $\beta|_S = \alpha$. □

Remark. A homomorphism $f: M \to N$ that is composite of homomorphisms $f = gh$ is said to factor through g and h. A homomorphism f factors uniquely through every epimorphism whose kernel is contained in that of f and through every monomorphism whose image contains the image of f.

3.2 Projective modules

Projection. Let M be an R-module. Let K be a direct summand of M with complementary direct summand K', that is, $M = K \oplus K'$. Then, $P_K : K \oplus K' \to K$, where $k \in K$, $k' \in K'$ defines an epimorphism $P_K : M \to K$ called the *projection* of M on K along K'.

Proposition 3.2.1. Let $M = K \oplus K'$ as given above, then, the projection of M on K along K' is the unique epimorphism $M \xrightarrow{P_k} K \longrightarrow 0$ which satisfies $P_K|_K = 1_K$ and Ker $P_K = K'$.

Proof: P_K satisfies the given conditions. Let $f : M \to K$ be such that $f|_K = 1_K$ and $Ker\ f = K'$, then, for all $k \in K$, $k' \in K'$, $f(K + K') = f(K) + f(K') = K + 0 = K = P_K(K + K')$. Again, if K is the direct summand of M with complementary direct summand K', $M = K \oplus K'$. Then, K' is a direct summand of M with K as its complementary direct summand. If P_K is the projection of M on K along K', then the projection P_K' of M on K' along K can be characterized by; $P_K' : m \mapsto m - P_K(m), m \in M$.

Theorem 3.2.2. Suppose that $i_K : K \to M$ and $i_K' : K' \to M$ represent inclusion mapping where $M = K \oplus K'$, then

$$0 \longrightarrow K' \xrightarrow{i_K'} M \xrightarrow{P_K} K \longrightarrow 0$$

$$0 \longrightarrow K \xrightarrow{i_K} M \xrightarrow{P_K} K \longrightarrow 0$$

are split exact.

Proposition 3.2.3 Let $M = K \oplus K'$. Let P_K be the projection of M on K along K', and let L be a submodule of M. Then, $M = L \oplus K'$ if and only if $P_K|_L : L \to K$ is an isomorphism.

Projective modules: A module P is called a *Projective module* if given any diagram

$$\begin{tikzcd} & M \arrow[d, "g"] \\ P \arrow[ur, "h"] \arrow[r, "f"'] & N \end{tikzcd}$$

where $g : M \to N$ is surjective and $f : P \to N$ is an R-module homomorphism, there exists a homomorphism $h : P \to M$ such that the diagram is commutative. $'h'$ is a lifting of f. In other words, given an epimorphism $g: M \to N$, then, any homomorphism $f: P \to N$ can be factored as $f = gh$ (f is lifted to h).

Theorem 3.2.4. A free module is projective.

Proof: Suppose that the R-module $'F'$ is free. Let $\alpha: K \to M$ be an R-epimorphism and

$\beta: F \to M$ be any R-homomorphism. Since F is free, it has a basis. Let S be a basis for F. $\alpha(K) = M$ and for each $s \in S$, there exists $y_s \in K$ such that $\alpha(y_s) = \beta(s)$. Define $\beta': F \to K$ by $\beta'(s) = y_s$. Then, $\alpha\beta' = \beta$. $\Rightarrow F$ is projective. □

Theorem 3.2.5. Every direct summand of a projective module over R is projective.

Proof: Let $M \oplus N$ be a projective module with M, a direct summand of $M \oplus N$. Let $\pi: M \oplus N \to M$ be a projection map, $\beta: A \twoheadrightarrow B$ be an epimorphism, $\delta: M \to B$ be any R-homomorphism; where $i: M \to M \oplus N$ is an inclusion map. $\delta \circ \pi: M \oplus N \to B$, $\delta \circ \pi \circ i = (\delta \circ \pi) \circ i = \delta \circ (\pi \circ i) = \delta$. Then, $\delta \circ \pi = \beta \circ \gamma \Rightarrow \delta \circ \pi \circ i = \beta \circ \gamma \circ i = \delta \Rightarrow \delta$ is a lifting to $\gamma \circ i \Rightarrow M$ is projective. □

Theorem 3.2.6. Let M be an R-module. Then, M is projective if and only if M is a direct summand of a free R-module.

Proof: Assume M is projective and F is a free module, there exists an exact sequence

$$0 \longrightarrow K \xrightarrow{\alpha} F \xrightarrow{\beta} M \longrightarrow 0$$

Let $1_M: M \to M$ be identity map. Then, there exists $\gamma: M \to F$ such that $\beta\gamma = 1_M$. $\Rightarrow$ $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ is a split short exact sequence. $\Rightarrow F \approxeq K \oplus M$.

Conversely, let $\pi: M \oplus N \to M$ be the projection map, $\beta: A \twoheadrightarrow B$ an R-epimorphism and $\delta: M \to B$ any R-homomorphism, then, $\delta\pi: M \circ N \to B$ lifts to $\gamma: M \circ N \to A$ since $M \circ N$ is free and is projective. $\delta\pi = \beta\gamma$. Let $i: M \to M \circ N$ be the inclusion map. Then, $\beta(\gamma i) = \delta(\pi i) = \delta$ $\Rightarrow \delta$ lifts to γi. $\Rightarrow M$ is projective. □

Theorem 3.2.7. A direct sum of projective modules is projective if and only if each summand is projective.

Proof: Let M be the direct sum of R-module M_i, $i = 1,2,\dots,n$. Suppose each M_i is projective and consider the diagram

$$\begin{tikzcd} & M \arrow[d, "\phi"] & \\ A \arrow[r, "\pi"'] & M \arrow[r] & 0 \end{tikzcd}$$

where the row is exact. Let $\phi|_{M_i} = \phi|_i$ be the restriction of ϕ to M_i. By projectivity of M_i, there exists $\varphi_i \in \mathrm{HOM}_R(M_i, A)$ with $\varphi_i \pi = \phi_i$. Let $\varphi = \sum_i \varphi_i \in HOM_R(M, A)$. Then, $\varphi\pi = \phi \Rightarrow M$ is projective.

Conversely, assume M is projective and we have the diagram with exact row,

$$\begin{tikzcd} & M_\lambda \arrow[d, "\phi_\lambda"] & \\ A \arrow[r, "\pi"'] & B \arrow[r] & 0 \end{tikzcd}$$

extend ϕ_λ to $\phi \in \mathrm{HOM}_R(M, B)$ by defining $\phi = \delta_{i\lambda}\phi_\lambda$. Let $\varphi\pi = \phi$ for some $\varphi \in HOM_R(M, A)$. Then the restriction $\varphi_\lambda = \varphi|_{M_\lambda} = HOM_R(M_\lambda, A)$ make the diagram to be commutative. Thus, $\pi\varphi_\lambda = \phi_\lambda \Rightarrow$ each summand of M is projective. □

Theorem 3.2.8. An R-module P is projective if and only if every short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ splits.

Proof: Assume P is projective. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be exact. Given the diagram below

$$\begin{tikzcd} & P \arrow[d, "1_P"] \\ N \arrow[r, "g"'] & P \end{tikzcd}$$

By hypothesis, we can fill in the diagram with $g': P \to N$ to obtain a commutative diagram. Thus, $gg' = 1_P$ and the short exact sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ splits.

Conversely, suppose the short exact sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ splits, then, P is a direct summand of N and hence, P is projective. □

Theorem 3.2.9. Let R be a commutative ring with identity; M and N, R-modules. Suppose $\alpha: N \to N'$, then

(a.) $\alpha_*: \mathrm{HOM}_R(M,N) \to \mathrm{HOM}_R(M,N')$ given by $\alpha_*(y) = \alpha y$ is an R-homomorphism.
(ii) $(y'y)_* = y'_* y_*$ if $y'y$ is defined.

(b.) Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence. Then,

$0 \longrightarrow \mathrm{HOM}_R(N,M') \longrightarrow \mathrm{HOM}_R(N,M) \longrightarrow \mathrm{HOM}_R(N,M'') \longrightarrow 0$ is exact.

Proof: (a.) Let $\alpha_*: N \to N'$ be a homomorphism. To show that $\alpha_*: \mathrm{HOM}_R(M,N) \to \mathrm{HOM}_R(M,N')$ given by $\alpha_*(y) = \alpha y$ is an R-homomorphism. Let $y \in \mathrm{HOM}_R(M,N)$, then for each y, there exists $\alpha y \in HOM_R(M,N')$. We claim that α_* is an R-homomorphism. $\alpha_*: \mathrm{HOM}_R(M,N) \to \mathrm{HOM}_R(M,N')$. For $y_1, y_2 \in \mathrm{HOM}_R(M,N)$ and $r_1, r_2 \in R$ and for all $m \in M$,

$f(r_1 y_1 + r_2 y_2)m = f[y_1(mr_1) + y_2(mr_2)] = f[y_1(mr_1)] + f[y_2(mr_2)] = r_1 f(y_1)(m) + r_2 f(y_2)(m)$

$\Rightarrow \alpha_*: \mathrm{HOM}_R(M,N) \to \mathrm{HOM}_R(M,N')$ is an R-homomorphism.

(ii.) α_* is epic if and only if for each $\bar{y} \in \mathrm{HOM}_R(M,N')$, there exists $y \in \mathrm{HOM}_R(M,N)$ such that $\alpha(y) = \bar{y}$. Let $\bar{y}\bar{y}' \in N$. Since α_* is an R-homomorphism, $\alpha_*(\bar{y},\bar{y}') = \alpha_*(\bar{y})\alpha_*(\bar{y}') \Rightarrow (yy')_* = y_* y'_*$ where $yy' \in N'$.

(b.) Given that $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ is exact. To show that

$$0 \longrightarrow \mathrm{HOM}_R(N,M') \xrightarrow{HOM_R(N,i)} \mathrm{HOM}_R(N,M) \xrightarrow{HOM_R(N,p)} \mathrm{HOM}_R(N,M'') \longrightarrow 0$$

is exact. Let $f \in \mathrm{HOM}_R(N,M'')$. There exists $g \in \mathrm{HOM}_R(N,M)$ such that $\mathrm{HOM}_R(N,P) = pg = f \Rightarrow \mathrm{HOM}_R(N,P)$ is surjective. Thus, the exactness of $0 \longrightarrow \mathrm{HOM}_R(N,M') \longrightarrow \mathrm{HOM}_R(N,M) \longrightarrow \mathrm{HOM}_R(N,M'') \longrightarrow 0$ is established. □

Theorem 3.2.10. An R-module P is projective if and only if for any exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of R-modules, the sequence

$0 \longrightarrow \mathrm{HOM}_R(P,M') \longrightarrow \mathrm{HOM}_R(P,M) \longrightarrow \mathrm{HOM}_R(P,M'') \longrightarrow 0$ is exact.

Proof: Let $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ be exact. Suppose P is projective, then given $f \in \mathrm{HOM}_R(P, M'')$, there exists $g \in \mathrm{HOM}_R(P, M)$ such that $\mathrm{HOM}_R(P,p)(g) = pg = f$. In this case, $\mathrm{HOM}_R(P,p)$ is surjective and we have the exactness of

$$0 \longrightarrow \mathrm{HOM}_R(P, M') \xrightarrow{HOM_R(P,i)} \mathrm{HOM}_R(P, M) \xrightarrow{HOM_R(P,p)} \mathrm{HOM}_R(P, M'') \longrightarrow 0.$$

The converse holds. Suppose $\mathrm{HOM}_R(P, -)$ is exact and $p: N \twoheadrightarrow \mathrm{M}$ (with $N \xrightarrow{p} M$). Let $\mathrm{Ker}\ p = K$. Then, we have the exact sequence $0 \longrightarrow K \xrightarrow{i} N \xrightarrow{p} P \longrightarrow 0$ where i is the injection map. Applying the exactness of $\mathrm{HOM}_R(P, -)$ we have that P is projective. □

Conclusion.

A module P is finitely generated and projective if and only if P is a direct summand of a free module with a finite base since if P is a direct summand of a free module F with a finite base, then P is projective and is also a homomorphic image of a free module F. Thus, P has a finite set of generator. Also, P being a finitely generated and projective module implies that we have an exact sequence $0 \longrightarrow P' \longrightarrow F \longrightarrow P \longrightarrow 0$ where F is free and has a finite base. P is projective implies that F is isomorphic to $P \oplus P'$, thus, P is a direct summand of a free module with finite base.

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