

Approximate Solution of the Dirichlet Problems in Polar Co-Ordinates

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Abstract: - This paper is concerned with the investigation of the Helmholtz type equation with the Dirichlet boundary conditions in polar co-ordinates. We present a numerical method for solving this equation and obtain the matrix form of equations. For our purpose define the mesh points in the $r - \theta$ plane by the points of intersection of the circles $r = ih, (i = 1, \dots, n)$ and the straight lines $\theta = j\delta\theta, j = 0, 1, 2, \dots$

Keywords: - finite difference, circular boundary, polar co-ordinates, Cartesian co-ordinates, curved boundary.

I. INTRODUCTION

Many physical problems involve solving elliptic equations with circular boundaries. Finite difference problems involving circular boundaries usually are solved more conveniently in polar co-ordinates than Cartesian co-ordinates. In this case, we first transform the rectangular coordinate system into the convenient polar or cylindrical coordinates. In the present paper, we consider Helmholtz equation

$$U_{xx} + U_{yy} + \lambda U = F(x, y)$$

With the Dirichlet boundary conditions on Ω , where λ is a positive real constant. The Helmholtz equation or reduced wave equation is an elliptic partial differential equation. It takes its name from the German physicist Hermann Helmholtz (1821-1894), a researcher in acoustics, electromagnetism, and physiology. This equation occurs when we are looking for mono frequency or time harmonic solutions for the wave equation.

A. S. Fokas introduced a new method for solving boundary value problems for linear and for integrable nonlinear PDEs [1]. Daniel ben-Avraham and Athanassios S. Fokas applied this method to the Helmholtz equation [2]. We want to solve this problem in polar co-ordinates. K. Mohseni and T. Colonius presented a numerical treatment of polar coordinate singularities [4]. There are other methods to solve pure problems in polar or cylindrical coordinates [6, 3]. In the next section, we present a finite difference scheme for solution of Helmholtz equation.

II. FINITE DIFFERENCE SCHEME

Let us consider E.q (1) on $\Omega = \{(x, y) | x^2 + y^2 < 1\}$ with the Dirichlet $U = g$ boundary conditions on Ω . Note that if $\lambda = 0$, the Helmholtz differential equation reduces to Laplace equation $U_{xx} + U_{yy} = F(x, y)$. In this paper, we only consider those solutions U of (1) which are defined and analytic in the real variables x, y for domain Ω in the plane R^2 . By using the polar coordinate transformation $x = r \cos \theta$ and $y = r \sin \theta$ where $r = (x^2 + y^2)^{1/2}$ and $\theta = \arctan \frac{y}{x}$, and setting $u(r, \theta) = U(x, y)$ and $f(r, \theta) = F(x, y)$ E.q (1) becomes:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \lambda u = f(r, \theta) \quad 0 < r < 1, 0 < \theta < 2\pi$$

For non-zero values of r there is no problem, but at $r = 0$ the right side appears to contain singularities. In this case, we can replace the polar co-ordinate form of equation by its Cartesian equivalent. In the present paper, we choose a grid which the grid points are in the $r - \theta$ plane as follow:

$$r_i = \frac{2i + 1}{2} \Delta r \quad i = 0, 1, \dots, n + 1$$

ordinate. Afterwards, we present an implicit scheme and obtain the matrix form of this equation. Finally, the obtained system can be solved by various methods.

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