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Inventory Model (M,T) With Quadratic Backorder Costs And Continuous Lead Time Series 1

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Abstract: - We have assumed in this paper that demand follows a normal distribution, and lead times follow a gamma distribution and backorder costs is quadratic.
The (M, T) Model an order is made to bring it to M at review time. The model is derived from the (nQ,R,T) model which at review time an integral multiple Q is ordered. After deriving the inventory costs for (nQ, R,T) we set Q 0 to obtain the (M,T) inventory costs by making use of the differentiation of the (nQ,R,T) model.
The (M,T) inventory costs for constant lead times is then averaged over the states of the lead times. Extensive use is made of the Bessel functions of imaginary argument.

Keywords: - *Continuous lead times, gamma distribution, normal distribution, quadratic backorder costs, Bessel function, inventory costs.*

I. INTRODUCTION

In deriving the (M,T) model in which an order is placed to bring inventory level to M, we make use of the (nQ,r,T) model in which the quantity ordered at review time is an integral multiple of Q, nQ n = 1,2,3…, We have derive the inventory costs for (nQ,R,T) and take the limit. Q 0 and setting R=M to obtain the (M,T) inventory costs.
We treating the backorder costs ($C_\beta(t)$ as quadratic for the length of time t, of the backorder $C_\beta(t) = b_1 + b_2 t + b_3 t^2$.
The dead time is assumed to follow a gamma distribution.
After deriving the inventory costs for (M,T) model with constant lead times, when then average the inventory costs order the states of the lead times.

II. LITERATURE REVIEW

Pektria (2012) used annual costs to derive an expression for the EOQ using price dependent demand in quadratic form. Bertsimas (1). in his paper "Probabilistic service level Guarantee in makes-to stock, considered both linear and quadratic inventory costs and backorder costs".
Nasir, Packnejad, and Afficoo (2012)., utilized EOQ model with non-linear holding cost.
Hadley and Whitin[(2)] extensively developed the model (M,T) for constant lead time and linear backorder costs.
Zipkin (2006) treats both fixed and random lead times and examines both stationary and limiting distributions under different assumptions.

(nQ,R,T) Model, Quadratic Backorder Costs

$C_\beta(t)$ is the backorder cost (nQ,R,T) stands for the model in which at review time, the inventory position or the amount on hand plus on order at review time is less than or equal to R and the quantity ordered is a multiple of

Q. demand D follows a normal distribution at lead time, $C_\beta(t)$ the expected backorder costs where t is the length of time of a backorder is given by $C_\beta(t) = b_1 + b_2 t + b_3 t^2$. Demand distribution is

$$g(\mu, \sigma^2 t) = \frac{1}{2\pi\sqrt{\sigma^2 t}} esp - \frac{1}{2}\left(\frac{x-\mu}{\sqrt{\sigma^2 t}}\right) - \infty \quad x < \infty \text{ ....(1)}$$

If the inventory position of the system is R + Y immediately after the review at time t, then the expected backorder costs at time t + L.

$$= \frac{1}{Q}\int_0^Q D\int_0^L \int_0^t \frac{C_\beta(t-z)}{\sqrt{\sigma^2 t}} g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdy$$

Similarly the expected backorder costs at time t + L+T

$$= \frac{1}{Q}\int_0^Q D\int_0^{L+T} \int_0^t C_\beta(t-z) g\left(\frac{R-Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdy$$

Noting that $C_\beta(t) = b_1 + b_2 t + b_3 t^2$

Which gives $= \frac{1}{Q}\int_0^Q D\int_0^L D\int_0^t \frac{(b_1 + b_2(t-z) + b_3(t-z))}{\sqrt{\sigma^2 t}} g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdy$

and $\frac{1}{Q}\int_0^Q D\int_0^{L+T} \int_0^t \frac{(b_1 + b_2(t-z) + b_3(t-z)^2)}{\sqrt{\sigma^2 t}} g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdy$ ....

Integrating the b_1 factor first and simplifying we have

$$\frac{b_1 \sigma^2 LD}{2DQ}\left(\left(1 + \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$\frac{b_1 \sigma^2 LD}{2DQ}\left(\left(1 + \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2\right) F\left(\frac{R+Q-L}{\sqrt{\sigma^2 L}}\right) - R\left(\frac{R+Q-L}{\sqrt{\sigma^2 L}}\right) g\left(\frac{R+Q-L}{\sqrt{\sigma^2 L}}\right)\right) \text{......}$$

Integrating the b_2 factor and simplifying we have

$$\frac{b_2}{Q}\left(\frac{D^2L^3}{6} - \frac{\sigma^2 R}{6D^3} - \frac{DL^2}{2} - \frac{\sigma^2 R^2}{4D^2} + \frac{\sigma^2 L^2}{4} + \frac{LR^2}{2} - \frac{R^3}{6D} - \frac{\sigma^6}{8D^4}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+ \frac{b_2}{Q}\left(\sqrt{\sigma^2 L}\left(\frac{DL^3}{6} - \frac{LR}{3} + \frac{R^2}{6D} + \frac{\sigma^2 L}{12D} + \frac{\sigma^2 R}{4D^2} + \frac{\sigma^4}{4D^3}\right) g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{\sigma^6}{8D^4 Q} esp\left(\frac{2DR}{\sigma^2}\right) F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

Integrating the b_3 factor and simplifying we have

$$\frac{b_3 D}{Q}\left(\frac{R^4 L^3}{12D^3} + \frac{\sigma^2 R^3}{6D^4} + \frac{\sigma^4 R^2}{4D^5} + \frac{\sigma^6 R}{4D^6} + \frac{\sigma^8}{8D^7} - \frac{L^2\sigma^2 R}{2D^2} + \frac{L^2 R^2}{2D} - \frac{RL^3}{3} + \frac{L^3\sigma^2}{3D} - \frac{R^3 L}{3D^2} + \frac{L^4 D}{12}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{b_3}{Q}.2\sqrt{\sigma^2 L}Dg\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{\sigma^2 RL}{24D^3} + \frac{\sigma^4}{24}\frac{R}{D^4} - \frac{R^2 L}{8D^2} - \frac{\sigma^2 L^2}{8D} + \frac{L^2 R}{8D} - \frac{L^3}{24} + \frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^4 R}{8D^5} + \frac{\sigma^6}{8D^6}\right)$$

$$- \frac{1}{8}\frac{\sigma^8}{D^6}\frac{b_3}{Q}\exp\left(\frac{2DR}{\sigma^2}\right) F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)$$

Putting the factors together we have the inventory costs C excluding the costs dependent in stockouts only

$$C = \frac{Rc}{T} + \frac{S.Pout.}{T} + hc\left(\frac{Q}{2} + R - DL - \frac{DT}{2}\right)$$

$$+ \frac{b_1}{QT}(G_1(R_1T + L) - G_1(R_1L) - G_1(R+Q, T+L) + G_1(R_1L)\ ]$$

$$+ \frac{hc + b_2}{QT}(G_2(R_1T + L) - G_2(R_1L) - G_2(R+Q, T+L) + G_2(R+Q, T+L) + G_2(R+Q, L)$$

$$+ \frac{b_3}{QT}(G_3(R_1T + L) - G_3(R_1L) - G_3(R+Q, T+L) + G_3(R+Q_1L)$$

$$+ \frac{s}{QT}(G_{01}(R_1T + L) - G_{10}(R_1L) - G_{10}(R+Q, T+L) + G_4(R+Q, L)\,) \ldots\ldots\ldots\ldots(2)$$

Where $G_1(R,L) = \frac{\sigma^2 L}{2}\left(\left(1 + \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)^2\right)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)\right) \ldots(3)$

$$G_2(R_1T) = \left(\frac{D^2T^3}{6} - \frac{\sigma^4 R}{4D^3} - \frac{DT^2R}{2} - \frac{\sigma^2R^2}{4D^2} + \frac{\sigma^2T^2}{4} + \frac{TR^2}{2} - \frac{\sigma^6}{8D^4} - \frac{R^3}{6D}\right)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+ \left(\frac{DT^{5/2}\sigma}{6} - \frac{\sigma T^{3/2}}{3} - \frac{\sigma T^{1/2}R^2}{6D} - \frac{\sigma^3 T^{3/2}}{12D} + \frac{\sigma^3 T^{1/2}R}{4D^2}\right)$$

$$+ \frac{\sigma^5 T^{1/2}}{4D^3}\left(g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) + \frac{\sigma^6}{8D^4}esp\left(\frac{2DR}{\sigma^2}\right)\right)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From

$$G_3(R_1T) = D\left(\frac{R^4}{12D^3} + \frac{\sigma^2R^3}{6D^4} + \frac{\sigma^3R^2}{4D^5} + \frac{\sigma^2R^2}{4D^2} + \frac{\sigma^6 R}{4D^5} + \frac{\sigma^8}{8D^7}\right)$$

$$-\left(\frac{\sigma^2R^2}{2D^2} + \frac{T^2R^2}{2D} - \frac{R^3}{3} + \frac{T^3}{3D} - \frac{R^3}{3D^2} + \frac{T^{4D}}{12}\right)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$-2\sqrt{\sigma^2 T}\,D\left(\frac{\sigma^2 RT}{24D^3} + \frac{\sigma^4 T}{24D^3} - \frac{R^2T}{8D^2} + \frac{\sigma^6}{8D^6}\right)g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$-\frac{1}{8}\frac{\sigma^8}{D^2}esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$G_4(R,T) = \left(\frac{(R-DT)^2}{2D} + \frac{\sigma^2 R}{2D^2} + \frac{\sigma^4}{4D^3}\right)F\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right)$$

$$+\left(\frac{\sqrt{\sigma^5 T}}{2}\left(T - \frac{\sigma^2}{D^2} - \frac{R}{D}\right)g\left(\frac{R-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^4}{4D^3}F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)esp\left(\frac{2DR}{\sigma^2}\right)\right)$$

Model (M,T) Continuous Lead Time and Quadratic backorder costs

We will derive model (M, T) for the quadractic costs from the (nQ, R, T) model derived above by taking its limit Q $\rightarrow$ 0 and setting R to M.

Differentiating with respect to R the costs

$$\frac{\partial G_1(R,T)}{dR} = -\left(\sqrt{\sigma^2(T+L)}\,g\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) - (R-DT)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)\right) \ldots\ldots\ldots (7)$$

and we set G_5 (R,T) such that differentiation is

$= -G_{11}\left(R_1 T + L\right)$

Similarly

Lim Q $\longrightarrow 0$

$$G_3\left(\frac{R+Q,L}{Q}\right) = -G_{12}(R,TL) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Where G_6 (R,T) =

$$\left(\frac{\sigma^4+2D^4T^2}{4D} + R\frac{\left(\sigma^4+2D^4T^2\right)}{2D} + \frac{R^2}{2D}\right)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) + \frac{1}{2}\left(\sigma T^{3/2} - \frac{\sigma^3 T^{1/2}}{D^2} - \frac{T^{1/2}R}{D}\right)$$

$$g\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sigma^4}{4D^3}esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right) \ldots\ldots\ldots\ldots(9)$$

Similarly

$$\lim_{Q\to 0} G_3\frac{(R+Q,L)}{Q} = -G_{13}(R,L) \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Where G_{13} (R, T) =

$$-D\left(\frac{R^3}{3D^3} + \frac{\sigma^2 R^2}{2D^4} + \frac{\sigma^4 R}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{\sigma^2 T^2}{2D^2} + \frac{T^2 M}{D} - \frac{T^3}{3} - \frac{R^2 T}{D^2}\right)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2 T}}g\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)\left(\frac{-2}{3}\frac{\sigma^2 RT^2}{D^2} + \frac{\sigma^2 T^3}{3D^4} + \frac{\sigma^2 R^2 T}{3D^3} - \frac{\sigma^4 RT}{2D^3}\right)$$

$$\left(+\frac{\sigma^4 T^2}{6D^3} + 8\frac{\sigma^6 T^2}{D^5}\right) + esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)\frac{\sigma^6}{4D^5} \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Similarly

$$\text{Lim } Q \longrightarrow 0 \quad G_4\frac{(R+Q,L)}{Q} = -R_0(R,L) \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

$$\text{Where } R_0(R,T) = \left(T - \frac{R}{D} - \frac{\sigma^2}{2D^2}\right)F\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)$$

$$+\frac{\sqrt{\sigma^2 T}}{D\sqrt{2\pi}}esp - \frac{1}{2}\left(\frac{R+DT}{\sqrt{\sigma^2 T}}\right)^2 + \frac{\sigma^2}{2D^2}esp\left(\frac{2D^2}{\sigma^2}\right)F\left(\frac{M+DT}{\sqrt{\sigma^2 T}}\right) \ldots\ldots\ldots\ldots\ldots(13)$$

The cost equations for model (M, T) excluding the cost of stockouts dependent only on the number of stockouts

$$C = \frac{R_1+S}{T} + hc\left(M - DL - \frac{DT}{2}\right) + \frac{b_1}{T}\left(G_{11}\left(M,T+L\right)\right)$$

$$-C_5(M,L)) + \frac{(b^2+hc)}{T}\left(G_6(M,T+L) - G_6(M,L + \frac{b_3}{T}\left(G_7\left(M,T+L\right) - G_7\left(M,L\right)\right)\right) \ldots \quad (14)$$

Let

$$G_8(R) = \int_0^\alpha H(L)G_5(R,L)dl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

$$G_9(R) = \int_0^\alpha H(L)G_6(R,L)dL \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$G_{10}(R) = \int_0^\alpha H(L)G_7(R,L)dL \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

$$G_{11}(R) = \int_0^\alpha H(L)G_5(R,T+L)dL \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

$$G_{12}(R) = \int_0^\alpha H(L)G_6(R,T+L)dL \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

$$G_{13}(R) = \int_0^\alpha H(L)G_7(R,T+L)dL \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

From equation (7)

$$G_5(M,L) = \sqrt{\sigma^2 L}\, g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right) - (M-DL)F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right) \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

Multiplying by H(L) where $H(L) = \alpha^k \frac{\exp tL^{k-1}}{)(k)}$

Hence $H(L)G_{11}(M,L) = \frac{\alpha^k \exp(-\alpha L)}{)(k)}\left[L^{k1/2} g\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right) - (ML^{k-1} - DL^k)F\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)\right] \ldots(22)$

Noting that from

$$\int_0^\alpha H(L)g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)\frac{1}{\sqrt{\sigma^2 L}}dl = \frac{\alpha^k}{\sigma\sqrt{2\pi}}\, esp\frac{\left(\frac{Dx}{\sigma^2}\right)}{)(k)}\left[2\left(\frac{x^2}{2\alpha + \sigma^2 + D^2}\right)^{1/2(k-1/2)}\right.$$

$$\left. K_{k-1/2}\left(\frac{x}{\sigma^2}(2\alpha\sigma^2 + D^2)^{1/2}\right)\right]$$

If K is an integer then

$$K_{k-1/2}(Z) = K_{1/2}(Z)\sum_{j=0}^{k-1}\frac{(k+j-1)!}{j!(k-j-1)!}(2Z)^{-j}$$

Where $K_{\frac{1}{2}}(Z) = \frac{\sqrt{\pi}}{\sqrt{2}}(Z)^{\frac{-1}{2}}\exp(-Z)$

Hence $K_{\frac{1}{2}}(Z) = \sqrt{\pi}\sum_{j=0}^{k-1}\frac{(k+j-1)!}{j!(k-j-1)!}(2Z)^{-j-\frac{1}{2}}\exp(-Z)$

Where $K_v\left(2\sqrt{BY}\right)$ denotes Bessel function of imaginary argument.

Letting $\theta^2 = 2\alpha\sigma^2 + D^2$

From equation

$$\int_0^\alpha H(L)F\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)dL$$

$$= \frac{\alpha^k}{\sigma\sqrt{2\pi}}\, esp\frac{\left(\frac{Dx}{\sigma^2}\right)}{)(k)}\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^2(k-z)!}\left[2D\left(\frac{x}{\theta}\right)^{k-z+1/2} K_{k-z+1/2}\left(\frac{x\theta}{\sigma^2}\right)\right]$$

$$+ 2x\left(\frac{x}{\theta}\right)^{k-z+1/2} K_{k-z+1/2}\left(\frac{x\theta}{\sigma^2}\right) \ldots\ldots\ldots\ldots(24)$$

Applying (23) and (24)

$$G_8(M) = \int_0^\alpha H(L)G_{11}(M,L)dL$$

$$G_8(M)=\frac{\alpha^k \sigma esp\left(\frac{DM}{\sigma^2}\right)}{\sqrt{2\pi}}\left[2\left(\frac{M}{\theta}\right)^{k+1/2}K_{k+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right]$$

$$-\frac{\alpha^k \sigma esp\left(\frac{DM}{\sigma^2}\right)}{2\sigma\sqrt{2\pi}\,)(k)}\left[M\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^2(k-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+1/2}K_{k+1/2}\left(\frac{M\theta}{\sigma^2}\right)+2M\left(\frac{M}{\theta}\right)^{k-z+1/2}K_{k-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right.\right]$$

$$+D\sum_{z=1}^{k+1}\frac{k!}{\alpha^z(k+1-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+3/2}K_{k-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)+2M\left(\frac{m}{\theta}\right)^{k-z+1/2}K_{k-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

From equation (9)

$$G_{12}(M,L)=\left(\frac{\sigma^4}{4D^3}+\frac{DL^2}{2}+\frac{M\sigma^2}{2D^2}-ML+\frac{M^2}{2D}\right)F\left(\frac{m-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{1}{2}\left(\sigma L^{3/2}-\frac{\sigma^3 L^{1/3}}{D^2}-\sigma\frac{L^{1/2}M}{D}\right)g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\frac{\sigma^4 esp\left(\frac{DM}{\sigma^2}\right)}{4D^3}zF\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)\quad\ldots\ldots(26)$$

Simplifying

$$G_6\ (M,L)=\left[\left(\frac{\sigma^4}{4D^3}+\frac{\sigma^2}{2D^2}+\frac{M^2}{2D}\right)-ML+\frac{DL^2}{2}\right]F\left(\frac{m+DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{1}{2}\left(-L^{1/2}\left(\frac{\sigma^3}{D^2}+\frac{\sigma M}{D}\right)+\sigma L^{3/2}\right)g\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\frac{\sigma^4}{4D^3}esp\left(\frac{2DM}{\sigma^2}\right)F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)\quad\ldots\ldots(27)$$

Multiplying by $H(L)=\frac{\alpha^k \exp-\alpha L}{)(k)}$

Hence

$$H(L)G_{12}(M,L)=\frac{esp(-\alpha L)\alpha^k}{)(k)}\left[\left(\frac{\sigma^4}{4D^3}+\frac{\sigma^2 M}{2D^2}+\frac{M^2}{2D}\right)\right]$$

$$L^{k-1}-ML^k+\frac{DL^{K+1}}{2}F\left(\frac{m+DL}{\sqrt{\sigma^2 L}}\right)-\frac{1}{2}\frac{\alpha^k\exp(-\alpha L)}{\sqrt{(k)}}$$

$$\left[\left(\frac{\sigma^3}{D^2}+\frac{\sigma M}{D}\right)L^{k-1/2}-\sigma L^{k+1/2}\right]g\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right)$$

$$-\frac{\sigma^4}{4D^3}esp\left(\frac{Dm}{\sigma^2}\right)F\left(\frac{m+DL}{\sqrt{\sigma^2 L}}\right)esp\,\frac{(-\alpha L)L^k\alpha^k}{)(k)}$$

Noting that

$$\int_0^{\alpha} H(L)F\left(\frac{x+DL}{\sqrt{\sigma^2 L}}\right) esp\left(\frac{Dx}{\sigma^2}\right) dL$$

$$= \frac{\alpha^k}{2\sigma)(k)} \cdot \frac{1}{\sqrt{2\pi}} \sum_{z=1}^{k} \frac{(k-1)!}{\alpha^2(k-z)!}\left[-20\left(\frac{k}{\theta}\right)^{k-z+1/2} K_{k-2+1/2}\left(\frac{x\theta}{\sigma^2}\right)\right]$$

$$+2x\left(\frac{x}{\theta}\right)^{k-z-1/2} K_{k-z-1/2}\left(\frac{x\theta}{\sigma^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

Where $\theta^2 = 2\alpha\sigma^2 + D^2$

Hence

$\int_0^{\alpha} H(L)G_{12}(M, L)dL$ applying (23), (24) (29)

We have

$$G_{15}(M) = \frac{\alpha^k esp\left(\frac{DM}{\sigma^2}\right)}{2\sigma\sqrt{2\pi})(k)}\left[\left(\frac{\sigma^4}{4D^3} + \frac{\sigma^2 M}{2D^2} + \frac{M^2}{2D}\right)\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^2(k-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+1/2} K_{k-z+1/2}\left(\frac{m\theta}{\sigma^2}\right)\right)\right]$$

$$2\left(\frac{M}{\theta}\right)^{K-Z+1/2} k_{K-Z+1/2}\left(\frac{M\theta}{\sigma^2}\right) - M\sum_{Z=1}^{K+1}\frac{K!}{\alpha^2(K+1-Z)!}$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-z+3/2} K_{k-z+3/2}\left(\frac{M\theta}{\sigma}\right) + 2M\left(\frac{M}{\theta}\right)^{k-z+1/2} K_{k-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+\frac{D}{2}\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^2(k+2-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+5/2} K_{k-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+2M\left(\frac{M}{\theta}\right)^{k-z+3/2} K_{k-z+5/2}\left(\frac{m\vartheta}{\sigma^2}\right) - \frac{\alpha^k esp\left(\frac{2M}{\sigma^2}\right)}{2\sqrt{2\pi})(k)}$$

$$\left[2\left(\frac{\sigma^3}{D^2} + \frac{\sigma M}{D}\right)\left(\frac{M}{\theta}\right)^{k+1/2} K_{k+1/2}\left(\frac{m\theta}{\sigma^2}\right) - \sigma\left(\frac{M}{\vartheta}\right)^{K+3/2} K_{k+3/2}\left(\frac{m\theta}{\sigma^2}\right)\right]$$

$$-\frac{\sigma^4 \alpha^k esp\left(\frac{DM}{\sigma^2}\right)}{\sqrt{2\pi}\, 4D^3 2\sigma)(k)}\left[\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^2(k-z)!}\right]$$

$$\left(2M\left(\frac{M}{\theta}\right)^{k-z+1/2} K_{k-z+1/2}\left(\frac{M\theta}{\sigma^2}\right) - 2D\left(\frac{M}{\theta}\right)^{k-z+1/2} K_{k-z+1/2}\left(\frac{M\theta}{\vartheta^2}\right)\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30)$$

Simplifying G_{13} (M, L) equation (3)

We have

$$G_{13}(M, L) = -D\left[\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4}{2D^5} + \frac{\sigma^6}{4D^6}\right)\right]$$

$$-\frac{M^2L}{D^2}+L^2\left(\frac{M}{D}-\frac{\sigma^2}{2D^2}\right)-\frac{L^3}{3})F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)$$

$$+\frac{D}{\sqrt{\sigma^2}}g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\left[\left(\frac{\sigma^2M^2}{3D^3}+\frac{\sigma^4M}{2D^4}\right)L^{1/2}\right]$$

$$+\frac{8\sigma^6}{D^5}L^{1/2}+L^{3/2}\left[\left(-\frac{2\sigma^2M}{3D^2}+\frac{\sigma^4}{6D^3}\right)+\frac{\sigma^2L^{5/2}}{3D}\right]$$

$$+\frac{\sigma^6}{4D^6}esp\left(\frac{2DM}{\sigma^2}\right)F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right) \quad \ldots\ldots(31)$$

Hence H(L)

$$H(L)G_{13}(M,L)=\frac{\alpha^k esp(-\alpha L)}{\Gamma(k)}\left[\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^2}{2D^4}+\frac{\sigma^4M}{2D^5}+\frac{\sigma^6}{4D^6}\right)\right]$$

$$-\frac{M^2L^k}{D^2}+L^{k+1}\left(\frac{M}{D}-\frac{\sigma^2}{2D^2}\right)-\frac{L^{k+2}}{3})$$

$$F\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)+\frac{1}{\sigma}g\left(\frac{M-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{\sigma^2M^2}{3D^3}+\frac{\sigma^4M}{2D^4}\right)L^{k-1/2}$$

$$+\frac{8\sigma^6}{D^5}L^{k-1/2}+L^{k+1/2}\left(\frac{\sigma^4}{6D^3}-\frac{2}{3}\frac{\sigma^2M}{D^2}\right)$$

$$+\frac{\sigma^2}{3D}L^{k+3/2}\ ]\frac{esp(-\alpha L)\alpha^k}{\Gamma(k)}+\frac{\sigma^6}{4D^6}esp\left(\frac{2DM}{\sigma^2}\right)$$

$$\frac{\alpha^k esp(-\alpha L)L^{k-1}}{\Gamma(k)}F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) \quad \ldots\ldots(32)$$

Hence integrating

$$G_{16}(M)=\int_0^\infty H(L)G_{13}(M,L)dL \quad \text{applying (23) (24) and (29)}$$

We have

$$G_{16}(M)=\frac{\alpha^k}{2\sigma}\frac{esp\left(\frac{DM}{\sigma^2}\right)}{\Gamma(k)\sqrt{2\pi}}\left[\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^3}{2D^4}+\frac{\sigma^4M}{4D}+\frac{\sigma^6}{4D^5}\right)\right]$$

$$\sum_{z=1}^{k}\frac{(k-1)!}{\alpha^z(k-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+1/2}K_{k-z-1/2}\left(\frac{m\vartheta}{\vartheta}\right)+2M\left(\frac{M}{\theta}\right)^{k-z-1/2}K_{k-z-1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)\Bigg)$$

$$-\frac{M^2}{D^2}\sum_{z=1}^{k-1}\frac{k!}{\alpha^2(k+1-2)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+3/2}K_{k-z+3/2}+2M\left(\frac{M}{D}\right)^{k-z+1/2}K_{k-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+\left(\frac{M}{D}-\frac{\sigma^2}{2D^2}\right)\sum_{z=1}^{k+2}\frac{(k+1)!}{\alpha^z(k+2-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-z+5/2}K_{k-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+2M\left(\frac{M}{\theta}\right)^{k-z+3/2}K_{k-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)\Bigg)-\frac{1}{3}\sum_{z=1}^{k+3}\frac{(k+1)!}{\alpha^z(k+3-z)!}$$

$$+2D\left(\frac{M}{\theta}\right)^{k-z+7/2}K_{k-z+7/2}\left(\frac{M\theta}{\sigma^2}\right)+2M\left(\frac{M}{\theta}\right)^{k-z+5/2}K_{k-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)$$

$$+\frac{esp\left(\frac{DM}{\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}\frac{\alpha^k}{\Gamma(k)}\left[2\left(\frac{\sigma^2M^2}{3D^3}+\frac{\sigma^4M}{2D^4}+\frac{8\sigma^6}{D^5}\right)\left(\frac{M}{\theta}\right)^{k+1/2}\right]$$

$$K_{k+1/2}\left(\frac{M\theta}{\sigma^2}\right)+2\left(\frac{\sigma^4}{6D^3}+\frac{2\sigma^2M}{3D^2}\right)\left(\frac{M}{\theta}\right)^{k+3/2}K_{k+3/2}\left(\frac{M\theta}{\sigma_2}\right)$$

$$+\frac{2\sigma^2}{3D}\left(\frac{M}{\theta}\right)^{k+5/2}K_{k+5/2}\left(\frac{M\theta}{\sigma^2}\right)\Bigg]$$

$$+\frac{esp\left(\frac{DM}{\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}\frac{\alpha^k}{\Gamma(k)}\cdot\frac{1}{\sqrt{2\pi}}\left[\sum_{z=1}^{k}\frac{(k+1)}{\alpha^z(k-z)!}\left(-2\left(\frac{M}{\theta}\right)^{k-z+1/2}\right)\right]$$

$$+2M\left(\frac{M}{\theta}\right)^{k-z-1/2}\Bigg)K_{k-z-1/2}\left(\frac{M\theta}{\sigma^2}\right) \quad \text{...(33)}$$

Using equation (3)

$$G_{16}(M,L+T)=\sigma(L+T)^{1/2}g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)-\left[(M-DT)-DL\right]$$

$$F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) \quad \text{...(34)}$$

Multiplying by H(L)

$$H(L)G_5(M,L+T)=\frac{\sigma\alpha^k esp(-\alpha L)L^{k-1}(L+T)^{1/2}}{\sqrt{2\pi}\,\Gamma(K)}g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)$$

$$-\frac{\alpha^k esp(-\alpha L)}{\Gamma(k)}\left[(M-DT)L^{k-1}-DL^k\right]F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) \quad \text{............................(35)}$$

Hence applying (23), (24) and (29)

Let

$$G_{11}(M,T)=\int_0^\infty H(L)G_5-(M,L+T)dL$$

$$=\frac{\sigma esp}{\sqrt{2\pi}\,\Gamma(k)}\left(\alpha T+\frac{DM}{\sigma^2}\right)\alpha^k\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\left(2\left(\frac{M}{\theta}\right)^{k-j+1/2}K_{k-j+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+\frac{\alpha^k}{2\sigma\sqrt{2\pi}}esp\frac{\left(\alpha T+\frac{DM}{\sigma^2}\right)}{\Gamma(k)}\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\sum_{z=1}^{k-j}\frac{(k-1-j)}{\alpha^z(k-j-z)!}$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+1/2}K_{k-j-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)-2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2}K_{k-j-z-1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$(M-DT)-D\sum_{j=0}^{k}(-T)^{j}\binom{k}{j}\sum_{z=1}^{k+1-j}\frac{(k-j)!}{\alpha^{z}(k+1-j-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+3/2}\right.$$

$$\left.K_{k-j-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)+2M\left(\frac{M}{\theta}\right)^{k-j-z+4/2}K_{k-j-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right]$$

Subsisting L+T for L and simplifying

From equation (31)

$$G_{13}(M,L+T)=\left[\left[\left(\frac{M^3}{3D^3}+\frac{\sigma^2M^2}{2D^4}+\frac{\sigma^4M}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{M^2T}{D^2}\right)\right]\right.$$

$$-\frac{M^2L}{D^2}+\sum_{z=0}^{2}\binom{2}{\iota}T^{\iota}L^{2-i}\left(\frac{M}{D}-\frac{\sigma^2}{2D^2}\right)$$

$$-\frac{1}{3}\sum_{\iota=0}^{3}\binom{3}{\iota}T^{\iota}L^{3-\iota}\left.\right]F\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)+\frac{1}{\sigma}g\left(\frac{M-DL}{\sqrt{\sigma^2(L+T)}}\right)$$

$$\left[\left(\frac{\sigma^2M^2}{3D^3}+\frac{\sigma^4M}{2D^4}+\frac{8\sigma^6}{D^5}\right)\right](L+T)^{1/2}$$

$$+\left(\frac{\sigma^4}{6D^3}-\frac{2\sigma^2M}{3D^2}\right)(T+L)^{3/2}+\sigma^2(L+T)^{1/2}\sum_{\iota=0}^{2}\binom{2}{\iota}T^{i}L^{2-i}\left.\right]$$

$$+\frac{\sigma^6}{4D^6}esp\left(\frac{2DM}{\sigma^2}\right)F\left(\frac{M+DL}{\sqrt{\sigma^2L}}\right)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

Multiplying by H(L) we have

$$H(L)G_{13}(M,L+T)=-\frac{\alpha^k esp(-\alpha L)}{\Gamma(k)}\left[\left(\frac{M^3}{3D^3}+\frac{\sigma^2}{2D^4}+\frac{\sigma^4M}{2D^5}+\frac{\sigma^6}{4D^6}-\frac{M^2T}{D^2}\right)L^{k-1}\right.$$

$$+\frac{M^2L^k}{D^2}+\sum_{i=0}^{2}\binom{2}{i}T^{i}L^{k+-i}$$

$$\left(\frac{M}{D}-\frac{\sigma^2}{2D^2}\right)-\frac{1}{3}\sum_{i=o}^{3}\binom{3}{i}T^{i}L^{k-i+2}\left.\right]F\left(\frac{M-DL}{\sqrt{\sigma^2L}}\right)$$

$$+\frac{\alpha^k}{\sigma}esp\frac{(-\alpha L)}{\Gamma(k)}(L+T)^{1/2}g\left(\frac{M+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)\left[\left(\frac{\sigma^2M^2}{3D^3}+\frac{\sigma^4M}{2D^4}+\frac{8\sigma^6}{D^5}\right)\right.$$

$$+\frac{\sigma^4T}{6D^3}-\frac{2\sigma^2MT}{3D^2}\left.\right]L^{k-1}+\left(\frac{\sigma^4}{6D^3}-\frac{2\sigma^2M}{3D^2}\right)L^k$$

$$+\sigma^2\sum_{i=0}^{2}\binom{2}{i}T^{i}L^{k-1+1}\left.\right]+\frac{\sigma^6}{4D^6}esp\left(\frac{2DM}{\sigma^2}\right)$$

$$\frac{\alpha^k esp(-\alpha L)L^{k-1}}{\Gamma(k)} F\left(\frac{M+DL}{\sqrt{\sigma^2 L}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

Applying (23, (24, (25)

$$G_{19}(M,T) = \int_0^{\alpha} H(L) G_{12}(M, L+T)$$

$$= \frac{\alpha^k esp(\alpha T + \frac{DM}{\sigma})}{\sqrt{2\pi}\Gamma(k) 2\sigma}\left[\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^4 M}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{M^2 T}{D^2}\right)\right]$$

$$\sum_{z=1}^{k-1}\binom{k-1}{j}^j \sum_{z=1}^{k-j} \frac{(k-1-z)!}{\alpha^2(k-j-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+1/2}\right)$$

$$K_{k-j-z+1/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2} K_{k-j-z-1/2}\left(\frac{M\theta}{\vartheta}\right)$$

$$+\frac{M^2}{D^2}\sum_{j=0}^{k}\binom{k}{j}(-T)^j \sum_{z=1}^{k+1-j} \frac{(k-j)}{\alpha^2(k+1-j-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+3/2} K_{k-j-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2} K_{k-j-z-1/2}\left(\frac{M\theta}{\sigma^2}\right)$$

$$+\left(\frac{M}{D} - \frac{\sigma^2}{2D^2}\right)\sum_{\iota=0}^{2}\binom{2}{t} T \sum_{j=0}^{k+1-\iota}(-T)^j \left(\frac{k+1-i}{j}\right)$$

$$+\sum_{z=1}^{k+2+i-j} \frac{(k+1-2-j)!}{\alpha^2(k+2-\iota-z)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+5/2} K_{k-j-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)\right)$$

$$+2M\left(\frac{M}{\theta}\right)^{k-j-z+3/2} K_{k-h-z+3/2}\left(\frac{M\theta}{\sigma^2}\right)$$

$$-\frac{1}{3}\sum_{\iota=0}^{3}\binom{3}{\iota} T^{\iota} \sum_{j=0}^{k+z-i}(-T)^{\iota}\binom{k-t+2}{j}^{k+3-i-j} \sum_{z=1} \frac{(k+2-1-j)!}{\alpha^2(k+3-i-z)!}$$

$$2D\left(\frac{M}{\theta}\right)^{k-j-z+7/2} K_{k-j-z+7/2}\left(\frac{M\theta}{\sigma^2}\right) + 2M\left(\frac{M}{\theta}\right)^{k-j-z+5/2} K_{k-j-z+5/2}\left(\frac{M\theta}{\sigma^2}\right)$$

$$+esp\left(\alpha T + \frac{DM}{\sigma^2}\right)\frac{\alpha^k}{\Gamma(k)}\left[\sum_{j-0}^{k-1}(-T)^j \left(\frac{k-1}{j}\right)\left(\frac{M^3}{3D^3} + \frac{\sigma^2 M^2}{2D^4} + \frac{\sigma^2 M}{2D^5} + \frac{\sigma^2}{4D^6} - \frac{M^2 T}{D^2}\right)\right]$$

$$+\frac{\sigma^2 M}{2D^5} + \frac{\sigma^6}{4D^6} - \frac{M^2 T}{D^2} - 2\left(\frac{M}{\theta}\right)^{k-j+1/2} K_{k-j+1/2}\left(\frac{m\vartheta}{\sigma^2}\right)$$

$$+2\left(\frac{\sigma^4}{6D^3} - \frac{2\sigma^2 M}{3D^2}\right)\sum_{j=0}^{k}(-T)^j \binom{k}{j}\left(\frac{M}{\theta}\right)^{k-i+3/2}$$

$$K_{k-j+3/2}\left(\frac{M\theta}{\sigma^2}\right) + 2\sigma^2 \sum_{i-0}^{2}(-T)\binom{k-i+1}{j}$$

$$\left(\frac{M}{\theta}\right)^{k-i-j+5/2} K_{k-i-j+5/2}\left(\frac{M\theta}{\sigma^2}\right) + \frac{\sigma^6}{4D^6}\frac{\alpha^k esp\left(\frac{DM}{\sigma^2}+\alpha T\right)}{\Gamma(k)2\sigma\sqrt{2\pi}}\sum_{j=0}^{k-1}\binom{k-1}{j}(-T)^j$$

$$\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\alpha^2(k-j-z)!}\left(-2D\left(\frac{M}{\theta}\right)^{k-j-z+1/2}K_{k-j-z+1/2}\left(\frac{M\theta}{\sigma^2}\right)\right)+2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2}K_{k-j-z-1/2}\left(\frac{M\theta}{\sigma^2}\right)$$

From equation (27) substituting L+T for L

$$G_{12}(M,_1L+T)=\left[\left(\frac{\sigma^4}{4D^3}+\frac{\sigma^2M}{2D^2}+\frac{M^2}{2D}-TM\right)-LM+\frac{D}{2}\sum_{j=0}^{2}\binom{2}{2}T^2L^{2-2}F\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)\right.$$

$$\left.+\frac{\sqrt{(L+T)}}{2}\left[-\left(\frac{\sigma^3}{D^2}+\frac{\sigma M}{D}-\sigma L\right)g\left(\frac{M-D(L+T)}{\sqrt{\sigma^2(L+T)}}\right)\right]\right]$$

$$-\frac{\sigma^4}{4D^3}esp\left(\frac{2DM}{\sigma^2}\right)F\left(\frac{M+D(L+T)}{\sqrt{\sigma^2(L+T)}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(40)$$

Multiplying by H(L) we have

$$H(L)G_{12}(M,_1L+T)=\left[\left(\frac{\sigma^4}{4D^3}+\frac{\sigma^2M}{2D^2}+\frac{M^2}{2D}-TM\right)-L^{k-1}\right]$$

$$-L^k+\frac{D}{2}\sum_{z=0}^{2}\frac{2}{\iota}T^2L^{k+1-\iota}F\left(\frac{M-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)$$

$$+\frac{1}{2}(L+T)^{1/2}\left[-\left(\frac{\sigma^3}{D^2}+\frac{\sigma M}{D}-\sigma T\right)L^{k-1}+\sigma L^k\right]$$

$$g\left(\frac{MD(T+L)}{\sigma^2(T+L)}\right)-\frac{\sigma^4}{4D^3}\alpha^k\ \frac{esp(-\alpha L)}{\Gamma(k)}esp\left(\frac{2DM}{\sigma^2}\right)F\left(\frac{M+DL}{\sqrt{\sigma^2L}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(41)$$

Let $G_{18}(M,T)=\int_0^\alpha H(L)G_{12}(M,T+L)dL$ and ……………………………………..(42)

Applying (23) (21) and (29)

We have

$$G_{18}(M,T)=\frac{\alpha^k\exp(\alpha T+\frac{DM}{\sigma^2})}{\sqrt{2\pi}\Gamma(k)\sqrt{2\pi}}\left[\left(\frac{\sigma^4}{4D^3}+\frac{\sigma^2M}{2D^2}+\frac{M^2}{D}-TM\right)\right]$$

$$\sum_{j=0}^{k-1}(-T)^j\binom{k-}{j}\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\alpha^2(k-j-z)!}$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+1/2}K_{k-j}\left(\frac{M\theta}{\theta}\right)+2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2}K_{k-j-z-1/2}\left(\frac{M\theta}{\theta}\right)\right)$$

$$-M\sum_{z=1}^{k}(-T)^j\binom{k}{j}\sum_{z=1}^{k+1-j}\frac{(k-j)!}{\alpha^2(k+1-j-z)!}$$

$$\left(2D\left(\frac{M}{\theta}\right)^{k-j-z+1/2}K_{k-j}{}^{\left(\frac{M\theta}{\theta}\right)}+2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2}K_{k-j-z-1/2}{}^{\left(\frac{M\theta}{\theta}\right)}\right)+\frac{D}{2}\sum_{i=0}^{2}\binom{2}{t}T^{2}\sum_{j=0}^{k+1-2}(-T)^{j}$$

$$\binom{k+1-2}{j}\sum_{z=\iota}^{k+2-\iota-j}\frac{(k+1-\iota-j)!}{\alpha^{2}(k+2-\iota-j)!}\left(2D\left(\frac{M}{\theta}\right)^{k-j-z-2+5/2}\right.$$

$$\left.K_{k-j-z-\iota+5/2}+2M\left(\frac{M}{\theta}\right)^{k-j-z-2+3/2}K_{k-j-z-2+3/2}{}^{\left(\frac{M\theta}{\sigma^{2}}\right)}\right)$$

$$+\frac{\alpha^{k}esp(\alpha T+\frac{DM}{\sigma^{2}})}{2\sqrt{2\pi})(k)}\left[-2\left(\frac{\sigma^{3}}{D^{2}}+\frac{\sigma M}{D}-\sigma T\right)\sum_{j=0}^{k-1}T^{j}\binom{k-1}{j}\right]$$

$$\left(\frac{M}{\theta}\right)^{k-j+1/2}K_{k-j+1/2}{}^{\left(\frac{M\theta}{\sigma^{2}}\right)}+\sigma\sum_{j=0}^{k}(-T)^{j}\binom{k}{j}\left(\frac{M}{\theta}\right)^{k-5+3/2}K_{k-j+3/2}{}^{\left(\frac{M\theta}{\sigma^{2}}\right)}$$

$$-\frac{\sigma^{4}}{4D^{3}}\frac{\alpha^{k}esp(\alpha T+\frac{DM}{\sigma^{2}})}{)(k)2k\sqrt{(2\pi)}}\sum_{j=0}^{k-1}(-T)^{j}\binom{k-1}{j}\sum_{z=1}^{k-1}\frac{(k-1-j)!}{\alpha^{2}(k-j-z)!}$$

$$\left(-2D\left(\frac{M}{\theta}\right)^{k-j-z+1/2}K_{k-j-z+1/2}{}^{\left(\frac{M\theta}{\sigma^{2}}\right)}+2M\left(\frac{M}{\theta}\right)^{k-j-z-1/2}K_{k-j-z+1/2}{}^{\left(\frac{M\theta}{\sigma^{2}}\right)}\right)\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

Hence averaging than inventory cost of (M, T) over the states of L from equation (14) excluding the stockout cost

$$C=\frac{Rc+s}{T}+hc\left(M-\frac{Dk}{\alpha}-\frac{DT}{2}\right)+\frac{b_{1}}{T}(G_{17}(M,T)-G_{14}(M))$$

$$+\frac{(hc+b_{2})}{T}(G_{18}(M,T)-G_{15}(M))+\frac{b_{3}}{T}(G_{19}(M,T)-G_{16}(M))\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

III. IMPACT OF STUDY

The study will enable industries, or organizations with thousands of items in their warehouses in different locations to express their backorder costs more accurately in non linear formulations. This would give more realistic inventory costs for holding items in various locations of the world.

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