American Journal of Engineering Research (AJER)
e-ISSN : 2320-0847 p-ISSN : 2320-0936
Volume-03, Issue-03, pp-51-55
www.ajer.org

Research Paper Open Access

I- Continuous Functions in Ideal Bitopological Spaces

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***Abstract:* -** In this paper we introduce and study the concepts of (i,j)-**I**- continuous, (i,j)-**I**- open and (i,j)-**I**- closed functions in Ideal Bitopological Spaces.
AMS Mathematics Subject Classification (2000): 54A10, 54A05, 54E55

***Key words and phrases:* -** *Ideal Bitopological spaces, (i,j)- **I**- open and (i,j)- **I**- closed sets, (i,j)- **I**- continuous and (i,j)- precontinuous functions.*

I. INTRODUCTION

In 1961 Kelly introduced the concept of bitopological spaces as an extension of topological spaces. A bitopological space (X, τ_1, τ_2) is a nonempty set X equipped with two topologies τ_1 and τ_2 [4]. The notion of ideal in topological spaces was studied by Kuratowski [5] and Vaidyanathaswamy [10]. An ideal **I** on a topological space (X, τ) is a nonempty collection of subsets of *X* which satisfies the conditions ***(i)*** A $\in$ **I** and B $\subset$ A then B $\in$ **I** and ***(ii)*** A $\in$ **I** and B $\in$ **I** then A $\cup$ B $\in$ **I**. An ideal topological space is a topological space (X, τ) with an ideal **I** on X, and is denoted by (X, τ, **I**). Given an ideal topological space (X, τ, **I**) if $\mathcal{P}$(X) is the set of all subsets of X, a set operator, (.)*:$\mathcal{P}$(X) $\rightarrow$ $\mathcal{P}$(X) is called the local function A^*(τ, **I**) (in short A^*) [10] of A with respect to the topology τ and ideal **I** defined as $A^* = \{x \in X \mid U \cap A \notin I, \forall\, U \in \tau$, where $x \in U\}$. A Kuratowski closure operator Cl* (.) for a topology τ^* (τ, **I**), called the *-topology, finer than τ, is defined by Cl*(A) = A $\cup$ A* [10].The collection {V\J : V $\in$ τ and J $\in$**I**} is a basis for τ^* [9]. A subset A of X is called **I**- open if A $\subseteq$ int(A*****) and **I**- closed if its complement is open. A subset A of X is called *-dense in itself, (resp. τ^*- closed, *- perfect) if A $\subseteq A^*$ (resp. $A^* \subseteq$ A, A = A^*) [2]. A subset A of X is called preopen [8] if A $\subseteq$ int(Cl(A)). The complement of a preopen set is called preclosed. Every**I**- open set is preopen, but the converse may not be true.

II. PRELIMINARIES

Definition 2.1: [6] A function f:(X, τ) $\rightarrow$ (Y, σ) is said to be precontinuous if the inverse image of every open set in Y is preopen in X

Definition 2.2: [7] A function f: (X, τ, **I**) $\rightarrow$ (Y, σ) is said to be **I** - continuous if for every V $\in$ σ, f^{-1}(V) is **I**- open in X

Definition 2.3: [1] A function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2) is said to be pairwise continuous if inverse image of every σ_i- open (resp. σ_j- open) set in Y is τ_i- open (resp. τ_j- open) in X

Definition 2.4: [3] An ideal bitopological space is a quadruple (X, τ_1, τ_2, **I**) where **I** is an ideal defined on a bitopological space (X, τ_1, τ_2)

Throughout this paper, $A^{\tau_i^*}$ {resp. $A^{\tau_j^*}$} denote the local function of a subset A of X with respect topology τ_i {resp. τ_j} and τ_i - Cl(A) (resp. τ_j - Cl(A)) and τ_i - int(A) {resp. τ_j - int(A)} denote the closure and interior of A with respect to topology τ_i (resp. τ_j)

Definition 2.5: [3] A subset A of an ideal bitopological space (X, τ_1, τ_2, **I**) is called (i,j)- preopen if $A \subseteq \tau_i$ - int (τ_j- Cl(A)) where ; i, j=1, 2, $i \neq j$

Definition 2.6: [3] A subset A of an ideal bitopological space (X, τ_1, τ_2, **I**) is called (i,j) **I** - open if $A \subseteq \tau_i$ - int($A^{\tau_j^*}$) where; i, j=1, 2, $i \neq j$

III. (I,J) -I-CONTINUOUS FUNCTIONS

Definition 3.1: A function f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) is said to be (i,j)-**I** - continuous if f^{-1}(V) is (i,j)-**I** - open in X for every σ_i -open set V in Y; i, j=1, 2, $i \neq j$

Definition 3.2: A function f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) is said to be (i,j)- precontinuous if f^{-1}(V) is (i,j)- preopen in X for every σ_i -open V in Y; i, j=1, 2, $i \neq j$.

Remark 3.1: Every (i,j)-**I** - continuous function is (i,j)- precontinuous but the converse is not true. For,
Example 3.1: Let X = {a, b, c, d} with topologies τ_1 ={X, ϕ, {a, c},{a, b, c}, {b}}; τ_2 = {X, ϕ, {a, b}, {a, b, d}, {d}} and **I** = {ϕ, {a}} be an ideal on X. Let Y = {p, q, r, s} with topologies σ_1 = {Y, ϕ, {p, r}, {p, q, r}, {q}}; σ_2 = {Y, ϕ, {p, q}, {p, q, s}, {s}}. Then f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) defined by f(a) = p, f(b) = q, f(c) = r, f(d) = s is (1,2)- precontinuous but not (1,2)-**I**- continuous because (p, r) is open and f^{-1}(p, r) is (1,2)- preopen but not (1,2)-**I**- open.

Definition 3.3: A function f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) is said to be pairwise **I**- continuous if f^{-1}(V) is τ_i-**I**- open (resp. τ_j-**I**- open) in X for every σ_i- open (resp. σ_j- open) set V in Y

Remark 3.2: The concepts of pairwise **I**- continuity and (i,j)-**I**- continuity are independent.

Example 3.2: Let X = {a, b, c, d} with topologies τ_1 ={X, ϕ, {a, c}, {a, b, c}, {b}}; τ_2 = {X, ϕ, {a, b}, {a, b, d}, {d}} and **I** = {ϕ, {a}} be an ideal on X. Let Y = {p, q, r, s} with topologies σ_1 = {Y, ϕ, {p, r}, {p, q, r}, {q}} σ_2 ={Y, ϕ, {p, q}, {p, q, s}, {s}}. Then f: (X, τ_1, τ_2, **I** $\rightarrow$ (Y, σ_1, σ_2) defined by f(a) = p, f(b) = q, f(c) = r, f(d) = s is pairwise **I**- continuous but not (1,2)-**I**- continuous, because (p, r) is open, f^{-1}(p, r) is τ_1-**I**- open, but not (1,2)-**I**- open in X.

Example 3.3: Let X = {a, b, c, d} with topologies τ_1 = {X, ϕ, {a, c}, {a, b, c}, {b}}; τ_2 = {X, ϕ, {a, b}, {a, b, d}, {d}} and **I** = {ϕ, {a}} be an ideal on X. Let Y = {p, q, r, s} with topologies σ_1 ={Y, ϕ, {p, r}, {p, q, r}, {q}} σ_2 ={Y, ϕ, {p, q}, {p, q, s}, {s}}. Then f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) defined by f(a) = p, f(b) = q, f(c) = r, f(d) = s is (1,2)-**I**- continuous but not pairwise **I**- continuous because (p, q, r) is open in Y, f^{-1}(p, q, r) is (1,2)-**I**- open but not τ_i-**I**- open in X.

Theorem 3.1: For a function f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) the following conditions are equivalent:
(a) f is (i,j)-**I**- continuous.
(b) For each $x \in X$ and each $V \in \sigma_i$ containing f(x), there exists an (i,j)-**I**- open set W in X such that $x \in W$ and $f(W) \subset V$.
(c) For each $x \in X$ and each $V \in \sigma_i$ containing f(x), $(f^{-1}(V))^{\tau_j^*}$ is a τ^i- neighborhood of x
Proof:
(a) $\Rightarrow$ ***(b)*** $V \in \sigma_i$ containing f(x). Hence by ***(a)***, f^{-1}(V) is (i,j)- **I**- open set in X containing x. Put W = f^{-1}(V) then $x \in W$ and $f(W) \subset V$.
(b) $\Rightarrow$ ***(c)*** Since $V \in \sigma_i$ containing f(x), then by ***(b),*** there exists an (i,j)- **I**- open set W in X containing x s.t. $f(W) \subset V$. So, $x \in W \subseteq (\tau_i\text{-}(\mathrm{int}(W)))^{\tau_j^*} \subseteq (\tau_i\text{-}\mathrm{int}(f^{-1}(V)))^{\tau_j^*} \subseteq (f^{-1}(V))^{\tau_j^*}$. Hence $(f^{-1}(V))^{\tau_j^*}$) is a τ_i - neighborhood of x.
(c) $\Rightarrow$(***a)*** Obvious.

Theorem 3.2: For a function f: (X, τ_1, τ_2, **I**) $\rightarrow$ (Y, σ_1, σ_2) the following conditions are equivalent:
(a) f is (i,j)- **I**- continuous

(b) The inverse image of each σ_j- closed set in Y is (i,j)- **I**- closed in X.

(c) τ_i - int $f^{-1}(M)^{\tau_j^*} \subset \tau_i$ - $(f^{-1}(M^{\sigma_i^*})$, for each σ_i^*-dense-in-itself subset M of Y.

(d) τ_i - $f(int(U))^{\tau_j^*} \subset \tau_i$ - $f(U)^{\tau_j^*}$, for each $U \subset X$, and for each σ_i^*- perfect subset of Y.

Proof:

(a) $\Rightarrow$ ***(b)*** Let $M \subset Y$ be σ_i- closed in Y, then Y\M is σ_i open in Y, then by ***(a)***, f^{-1}(Y\M) = X\f^{-1}(M) is (i,j)- **I**- open in X. Thus, f^{-1}(M) is (i,j)-**I**- closed in X.

(b) $\Rightarrow$ ***(c)*** Let $M \subset Y$ be σ_i- closed in Y, Since $M^{\sigma_i^*}$ is also σ_i- closed in Y, then by ***(b)*** $f^{-1}(M^{\sigma_i^*})$ is (i,j)- **I**- closed in X. Next, by using Theorem 2.4 [6], τ_i - int $f^{-1}(M^{\sigma_i^*})^{\tau_j^*} \subset f^{-1}(M^{\sigma_i^*})$ and since $M^{\sigma_i^*}$ is σ_i^*- dense in itself, τ_i - int $f^{-1}(M)^{\tau_j^*} \subset \tau_i$ - int $f^{-1}(M^{\sigma_i^*})^{\tau_j^*} \subset (f^{-1}(M^{\sigma_i^*})$.

(c) $\Rightarrow$ ***(d)*** Let $U \subset X$ and W = f(U), then by ***(c)***, τ_i - $int(U)^{\tau_j^*} \subset \tau_i$ - int $f^{-1}(W)^{\tau_j^*} \subset \tau_i$ - int $f^{-1}(W^{\sigma_i^*})^{\tau_j^*} \subset (f^{-1}(W)^{\tau_j^*}$ (because $W^{\sigma_i^*}$ is perfect). Hence, τ_i - f(int $(U)^{\tau_j^*} \subset \tau_i$ - $(W)^{\tau_j^*} = \tau_i$ - $f(U)^{\tau_j^*}$.

(d) $\Rightarrow$ ***(a)*** Let $V \in \sigma_i$,W = Y\V, and $U \subset f^{-1}(W)$, then $f(U) \subset W$ and by ***(d)***, τ_i - $f(int(U))^{\tau_j^*} \subset \tau_i$ - $f(U)^{\tau_j^*} \subset \tau_i$ - $(W)^{\tau_j^*}$ = W (because W is σ_i^*- perfect). Thus, τ_i - $(int(f^{-1}(W)))^{\tau_j^*} = \tau_i$ - $(int(U))^{\tau_j^*} \subset f^{-1}(W)$, and therefore, f^{-1}(Y\V) is (i,j)-**I**- closed. Hence, f^{-1}(V) is (i,j)-**I**- open in X and f is (i,j)-**I**- continuous.

Theorem 3.4: Let f: $(X, \tau_1, \tau_2, \mathbf{I}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)- **I**- continuous and $U \in \tau_1 \cap \tau_2$. Then the restriction f\U is (i,j)-**I**- continuous.

Proof:

Let $V \in \sigma_i$. Then, τ_i - $f^{-1}(V)^{\tau_j^*} \subseteq \tau_i$ - $int(f^{-1}(V))^{\tau_j^*}$ and so $U \cap \tau_i$ - $f^{-1}(V)^{\tau_j^*} \subseteq U \cap \tau_i$ - $int(f^{-1}(V))^{\tau_j^*}$ Thus $(f\backslash U)^{-1}(V) \subseteq U \cap \tau_i$ - $int(f^{-1}(V))^{\tau_j^*}$. Since $U \in \tau_i$, we get $(f\backslash U)^{-1}(V) = \tau_i$ - $int(U \cap (f^{-1}(V))^{\tau_j^*}$ [5] $\subseteq \tau_i$ - $int(U \cap f^{-1}(V))^{\tau_j^*} = \tau_i$ - $int((f\backslash U)^{-1}(V))^{\tau_j^*}$. Hence $(f\backslash U)^{-1}(V)$ is (i,j)**I**- open and f\U is (i,j)- **I**- continuous.

Theorem 3.5: For a function f: $(X, \tau_1, \tau_2, \mathbf{I}) \to (Y, \sigma_1, \sigma_2)$ and $\{U_\alpha: \alpha \in \Delta\}$ be a biopen cover of X. If the restriction function $f\backslash U_\alpha$ is (i,j)- **I** - continuous, for each $\alpha \in \Delta$, then f is (i,j)- **I**- continuous.

Proof:

Similar to Theorem 1.4

Theorems that follow are immediate and their obvious proofs have been omitted

Theorem 3.6: Let f: $(X, \tau_1, \tau_2, \mathbf{I}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)- **I**- continuous and a biopen function, then the inverse image of each open set in Y, which is (i,j)**I**- open set in X is also (i,j)- preopen in X.

Theorem 3.7: Let f: $(X, \tau_1, \tau_2, \mathbf{I}) \to (Y, \sigma_1, \sigma_2)$ is (i,j)- **I**- continuous and τ_i- $f^{-1}(V^{\tau_j^*}) \subset \tau_i$ - $(f^{-1}(V))^{\tau_j^*}$, for each $V \in \sigma_i \subset Y$. Then the inverse image of each (i,j)- **I**- open set is (i,j)- **I**- open.

Remark 3.3: Composition of two (i,j)- **I**- continuous functions need not be (i,j)- **I**- continuous, in general, as shown by the following example.

Example 3.4: Let X = {a, b, c} with topologies $\tau_1 = \{X, \phi, \{a\}\}$, $\tau_2 = \{X, \phi\}$, and $\mathbf{I} = \{\phi, \{c\}\}$ be an ideal on X; Y = {a, b, c, d} with topologies $\sigma_1 = \{Y, \phi, \{a, c\}\}$ $\sigma_2 = \{Y, \phi\}$ and $\mathbf{J} = \{\phi, \{a\}\}$ be an ideal on Y; Z = {a, b, c} $\eta_1 = \{Z, \phi, \{c\}, \{b, c\}\}$ $\eta_2 = \{Z, \phi\}$. Let f: $(X, \tau_1, \tau_2, \mathbf{I}) \to (Y, \sigma_1, \sigma_2, \mathbf{J})$ be the identity function and let g: $(Y, \sigma_1, \sigma_2, \mathbf{J}) \to (Z, \eta_1, \eta_2)$ be defined as g(a) = a, g(b) = g(d) = b, g(c) = c. It is clear that both f and g are (i,j)-**I**- continuous but the composition function gof is not (i,j)**I**- continuous, because {c} is open, but $(gof)^{-1}$\{c\} = {c}is not (i,j)-**I**- open.

Theorem 3.8: For the functions f: $(X, \tau_1, \tau_2, \mathbf{I}) \to (Y, \sigma_1, \sigma_2)$ and g: $(Y, \sigma_1, \sigma_2, \mathbf{J}) \to (Z, \eta_1, \eta_2)$ if f is (i,j)-**I**- continuous and g is pairwise continuous then gof is (i,j)-**I**- continuous,.

Proof: Obvious.

IV. (I,J) I- OPEN AND (I,J) I- CLOSED FUNCTION

Definition 4.1: A function f: $(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2, \mathbf{J})$; i, j = 1, 2, $i \neq j$ is called (i,j)-**I**- open function (resp. (i,j)-**I**- closed function) if for each $U \in \tau_i$ (resp. $U \in \tau_i^c$), f(U) is an (i,j)**I**- open set in Y (resp. (i,j)-**I**- closed set in Y

Remark 4.1: (i,j)-**I**- open (resp. (i,j)-**I**- closed) function $\Rightarrow$ preopen (resp. preclosed) function but the converse is not true.

Example 4.1: Let X = Y = {a, b, c} with topologies τ_1 ={X, ϕ, {b}, {a, b}, {b, c}} τ_2 = {X, ϕ} the discrete topology; σ_1 = {Y, ϕ, {a}, {a, b}}; σ_2 = {Y, ϕ} the discrete topology and **J** = {ϕ, {a}, {b}, {a, b}} an ideal on Y. Then the identity function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) is preopen but not (1,2)-**I**- open, because {b} is open, but f(b) is a (2,1)-**I**- preopen set but not a (2,1)-**I**- open set.

Remark 4.2: The concepts of (i,j)-**I**- open functions and pairwise open functions are independent concepts

Example 4.2: Let X = Y = {a, b, c, d} with topologies τ_1 = {X, ϕ, {a, b}, {a, b, d}} τ_2 the discrete topology; σ_1 = {Y, ϕ, {a, b}, {a, b, c}}; σ_2 the discrete topology and **J** = {ϕ, {c}, {d}, {c, d}} an ideal on Y. Then the identity function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) is a (1,2)**I**- open function but not pairwise open function.

Example 4.3: Let X = Y = {a, b, c}; τ_1 ={X, ϕ, {a}}; σ_1 = {Y, ϕ, {a}, {a, b}}; τ_2 and σ_2 the respective discrete topologies on X and Y and **J** = {ϕ, {a}} an ideal on Y. Then the identity function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) is an open function but not a (1,2)-**I**- open function because, {a} is open, but f(a) = a is σ_2 open but not a (1,2)-**I**- open.

Theorems that follow are immediate and their proofs obvious from the definitions

Theorem 4.1: Let f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) be a function. Then the following are equivalent:

a) f is a (i,j)-**I**- open function.

b) For each x ϵ X and each neighborhood U of x, there exists an (i,j)-**I**- open set W $\subset$ Y containing f(x) such that W $\subset$ f(U)

Theorem 4.2: Let f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) be an (i,j)**I**- open function (resp. (i,j)-**I**- closed function) if W $\subset$ Y, and F $\subset$ X is a closed (resp. open) set containing f^{-1}(W), then there exists an (i,j)**I**- closed (resp. (i,j)-**I**- open) set H containing W such that f^{-1}(H) $\subset$ F

Theorem 4.3: If function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) is (i,j)-**I**- open, then τ_i- $f^{-1}(\mathrm{int}(B))^{\sigma_i^*} \subset$ τ_i- $(f^{-1}(B)^{\sigma_i^*}$ such that f^{-1}(B) is σ_1^* dense-in-itself, for every B $\subset$ Y

Theorem 4.4: For any one-one onto function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) the following are equivalent:

a) f^{-1} (Y, σ_1, σ_2, **J**) $\rightarrow$ (X, τ_1, τ_2) is (i,j)-**I**- continuous

b) f is (i,j)-**I**- closed

Theorem 4.5: If function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) is (i,j)-**I**- open and for each A $\subset$ X, σ_i- f(A)$^{\tau_j^*}$ $\subset$ σ_i- [f(A)]$^{\tau_j^*}$, then the image of each (i,j)-**I**- open set is (i,j)-**I**- open.

Theorem 4.6: Let function f: (X, τ_1, τ_2) $\rightarrow$ (Y, σ_1, σ_2, **J**) and g: (Y, σ_1, σ_2) $\rightarrow$ (Z, η_1, η_2, **K**) be two functions, where **I**, **J** and **K** are ideals on X, Y and Z respectively, then

a) If f is open and g is (i,j)-**I**- open then gof is (i,j)-**I**- open

b) f is (i,j)-**I**- open if gof is open; g is one-one and (i,j)-**I**- continuous

c) If f and g are (i,j)-**I**- open; f is surjective and $g(V)^{\sigma_i^*} \subset [g(V)]^{\sigma_i^*}$ for each V $\subset$ Y, then gof is (i,j)-**I**- open

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