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An Interpolation Process on the Roots of Hermite Polynomials on Infinite Interval

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ABSTRACT: For given arbitrary numbers $d_k, k = 1(1)n-1, d_k^*, k = l(1)n-1$ and $d_k^{**}, k = 1(1)n$, we seek to determine explicity polynomials $R_n(x)$ of degree atmost 3n-3 (n even), given by:

(1) $$R_n(x) = \sum_{k=1}^{n-1} d_k U_k(x) + \sum_{k=1}^{n-1} d_k^* V_k(x) + \sum_{k=1}^{n} d_k^{**} W_k(x),$$

Such that

$$R_n(y_k) = d_k, \quad k = 1(1)n-1,$$
$$R_n^{'}(y_k) = d_k^*, \quad k - 1(1)n-1$$
and
$$R_n(x_k) = d_k^{**}, k = 1(1)n,$$

where $\{x_k\}_{k=1}^{n}$ are the zeros of n^{th} Hermite polynomial $H_n(x)$ and $\{y_k\}_{k=1}^{n-1}$ are the zeros of $H_n^{'}(x)$.
Let the interpolated function f be continuously differentiable satisfying the conditions:

$$\overset{Lim}{|x| \to +\infty} x^{2\nu} f(x) \rho(x) = 0, \quad \gamma = 0, 1, 2., \ldots..$$
and
$$\overset{Lim}{|x| \to +\infty} x^{2\nu} \rho(x) f'(x) = 0, \quad \text{where } \rho(x) = e^{-x^2/2},$$

further in (1) $d_x = f(y_k), \; k = (1)n-1,$

$$d_k^* = f'(y_k), \delta\kappa = \circ\left(e^{\delta y^2 k} \omega(f', \frac{1}{\sqrt{n}})\right), k = 1(1)n-1, \circ < \delta < 1$$

$d_k^{**} = f'(x_k), k = 1(1)n,$ then for the sequence of inter polynomials $R_n(n = 2, 4, \ldots)$, we have the estimate

$$e^{-\nu x^2} \left| f(x) - R_n(f, x) \right| = 0(\omega(f'; \frac{1}{\sqrt{n}}) \log n), \quad \nu > \frac{3}{2}$$

Which holds the whole real line, O does not depend on n and x and ω is the modulus of continuity of f' introduce by G. Freud.

Keywords: Hermite, Interpolation

I. INTRODUCTION

Earlier Pal [8] proved that when function values are prescribed on one set of n points and derivative values on another set of $n-1$ points, then there exists no unique polynomial of degree $\leq 2n-1$, but prescribed function value at one more point not belonging to the former set of n points there exists a unique polynomial of degree $\leq 2n-1$. Eneduanya [2] proved its convergence on the roots of $\pi_n(x)$.

Let $$-\infty < x_{n,n} < x^*_{n-1} < \ldots < x^*_{2,n} < x_{1,n} < \infty$$

be a given system of $(2n-1)$ distinct points. $L\cdot$Szili [11] determined a unique polynomial R_n lowest possible degree 2n-1 (for n even) given by:

$$R_n(x) = \sum_{i=1}^{n} Y_{i,n} A_{i,n}(x) + \sum_{i=1}^{n-1} Y^*_{1,n} B_{i,n}(x),$$

satisfies the conditions:

$$R_n(x_{i,n}) = Y_{i,n}, \quad \text{i} = 1,2,\ldots n$$

$$R'_n(X^*_{v,n}) = Y'_{v,n}, \quad v = 1,2,\ldots n-1$$

and

$$R_n(0) = 0$$

If the interpolated function f is continuously differential

$$f(0) = 0 \text{ and } \lim_{|x|\to\infty} e^{-x^2/2} x^{2v} f(x) = 0, \quad v = 0,1,2,\ldots$$

$\lim_{|x|\to\infty} f'(x) e^{-x^2/2} = 0$, then the sequence $\{R_n(x)\}$ satisfies the relation

$$e^{-vx^2}\left|f(x) - R_n(x)\right| = 0(\omega(f', \frac{1}{\sqrt{2}})\log n), \quad v > 1$$

which holds on the whole real line and 0 does not depend or n and x.

Further K.K. Mathur and R.B. Saxena [6] extended the results of L. Szili to the case of weighted (0,1,3,)-interpolation on Infinite interval.

In this paper, we consider a special problem of mixed type, (0,1;0)-interpolation on the zeros of Hermite polynomial

(1.1) Let $\{X_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$ be the zeros of $H_n(x)$ and $H_n'(x)$, where

The fundamental polynomials of Lagrange interpolation are given by

(1.2) $$1_k(x) = \frac{H_n(x)}{H'_n(x_k)(x-x_k)}, \quad k = 1(1)n.$$

and

(1.3) $$L_k(x) = \frac{H'_n(x)}{H'_n{}'(y_k)(x-y_k)}, \quad k = 1(1)n-1$$

In this paper, study the following:

<u>$(0,1:0)$ – Interpolation on Infinite Interval.</u>

Let n be even, then for given arbitrary sequence of numbers $\{d_k\}_{k=1}^{n-1}, \{d_k^*\}_{k=1}^{n-1}$ and $\{d_k^{**}\}_{k=1}^{n}$, there exists a unique polynomial $R_n(x)$ of degree $\leq 3n-3$, such that

(1.4)
$$\begin{cases} R_n(y_k) = d_k, & k = 1(1)n-1 \\ R_n'(y_k) = d_k^*, & k = 1(1)n-1 \\ and & \\ ,R_n(y_k) = d_k^{**}, & k = 1(1)n \end{cases}$$

For n odd, $R_n(x)$ does not exist uniquely. Precisely we shall prove the following:

Theorem 1:

For n even,

(1.5)
$$R_n(x) = \sum_{k=1}^{n-1} d_k U_k(x) + \sum_{k=1}^{n-1} d_k^* V_k(x) + \sum_{k=1}^{n} d_k^{**} W_k(x),$$

where $U_k(x)$, $k = 1(1)n-1$ and $W_k(x)$, $k = 1(1)n$ are the fundamental polynomial of the first kind and $V_k(x)$, $k = 1(1)n-1$ are the fundamental polynomials of the second king of mixed type $(0,1:0)$ interpolation. Each such fundamental polynomial is of degree at most $3n-3$, given by:

(1.6)
$$U_k(x) = \frac{H_n(x) L_k^2(x)\left[1 - 2y_k(x - y_k)\right]}{H_n(y_k)}, \quad k = 1(1)n-1$$

(1.7)
$$V_k(x) = \frac{H_n(x) H_n'(x) L_k(x)}{H_n(y_k) H_n''(y_k)}, \quad k = 1(1)n-1$$

and

(1.8)
$$W_k(x) = \frac{H_n'^2(x) 1_k(x)}{H_n'^2(H_k)}, \quad k = 1(1)n.$$

where $1_k(x)$ and $L_k(x)$ are given by (1.2) and (1.3) respectively

Theorem 2:

Let the interpolated function $f : R \to R$ be continuous differentiable, such that

(1.9)
$$\begin{cases} \underset{|x| \to +\infty}{Lim} x^{2k} f(x) \rho(x) = 0 \quad (k = 0,1,\ldots) \\ and \\ \underset{|x| \to +\infty}{Lim} \rho(x) f'(x) = 0, \text{ where } \rho(x) = e^{-\beta x^2}, \circ \leq \beta < 1. \end{cases}$$

Further, taking the numbers δ_k as:

(1.10) $$\delta_k = 0\left(e^{\delta y_k^2}\right) w\left(f';\frac{1}{\sqrt{n}}\right), \ k = 1(1)n-1, \ \circ < \delta < 1,$$

where w is the modulus of continuity of f', then

(1.11) $$R_n(f,x) = \sum_{k=1}^{n-1} f(y_k)U_k(x) + \sum_{k=1}^{n-1} \delta_k V_k(x) + \sum_{k=1}^{n} f(x_k)W_k(x)$$

satisfies the relation:

$$e^{-vx^2}\left|f(x) - R_n(x)\right| = 0\left(\log n \ w\left(f';\frac{1}{\sqrt{n}}\right)\right), \quad v > \frac{3}{2}.$$

which holds on the whole real line and 0 does not depend on n and x.

Remark.

$w(f,\delta)$ denotes the special form of modulus of continuity introduced by G. Freud [3] given by:

(1.12) $$w(f,\delta) = \underset{\circ \le t \le \delta}{Sup} \left\|W(x+t)f(X+t) - W(x)f(x)\right\| + \left\|T(\delta x)W(x)\right\|$$

where

$$T(x) = \begin{cases} |x|, & \text{for } |x| \le 1 \\ 1, & \text{for } |x| > 1 \end{cases}.$$

and $\|\bullet\|$ denotes the sup-norm in $C(R)$. If $f \in C(R)$ and $\lim_{|x| \to \infty} W(x) f(x) = 0$, then $\lim_{\delta \to \circ} w(f,\delta) = 0$.

II. PRELIMINARIES.

In this section, we shall give some well known result which we shall use in the sequel.

The differential equation satisfied by $H_n(x)$ is given by:

(2.1) $$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

(2.2) $$H_n'(x) = 2nH_{n-1}(x).$$

From (1.2), we have

(2.3) $$1_K(x_j) = \begin{cases} 0 & j \ne k \\ & \text{for} \\ 1 & j = k \end{cases}, \quad k = 1(1)n$$

(2.4) $$1'_k(x_j) = \begin{cases} \dfrac{H'_n(x_j)}{H'_n(x_k)(x_j - x_k)}, & j \neq k \\ x_k, & j = k. \end{cases}$$

Form (1.3), one has

(2.5) $$L_k(y_j) = \begin{cases} 0 & j \neq k \\ & \text{for} \quad , \quad k = 1(1)n-1 \\ 1 & j = k \end{cases}$$

(2.6) $$x_k^2 \square \frac{k^2}{n}$$

(2.7) $$H_n(x) = 0\left\{n^{-1/4}\sqrt{2^n n!}\left(1 + 3\sqrt{|x|}e^{x^{2/2}}\right)\right\}, x \in R$$

(2.8) $$H'_n(x) \geq c2^{2+1}\left[\frac{n}{2}\right]!e^{\delta x_k^2}, 0 < \delta < 1.$$

(2.9) $$\sum_{i=0}^{n-1} \frac{H_i(y)H_i(x)}{2^i i!} = \frac{H_n(y)H_{n-1}(x) - H_{n-1}(y)H_n(x)}{2^n(n-1)!(y-x)}$$

From (1.2) and (2.9) at $y = x_k$, we have

(2.10) $$\left|1_k(x)\right| = \frac{0(1)2^{n+}n!\sqrt{n}e^{\frac{v_1}{2}(x^2+x_k^2)}}{H_n'^2(x_k)}, \quad v_1 > 1$$

(2.11) $$\sum_{k=1}^{n} e^{-\in x_k^2} 0\left(\sqrt{n}\right), \quad \text{where} \quad \in > 0$$

(2.12) $$\sum_{k=1}^{n} e^{\delta x_k^2}\left(H'_n(x_k)\right)^{-2} = 0\left(2^{n+1}n!\right)^{-1}, \quad 0 < \delta < 1$$

and

(2.13) $$\frac{2^n\left[\left(\frac{n}{2}\right):\right]^2}{(n+1):} \square n^{-1/2}, \quad n = 1,2,\ldots$$

III. PROOF OF THEOREM 1.

Using the results given in preliminaries and a little computation, one can easily see that the polynomials given (1.6), (1.7) and (1.8) satisfy the conditions:

For $k = 1(1)n-1$

(3.1)
$$\begin{cases} U_k(y_j) = \begin{cases} 0 & j \neq k \\ & \text{for} \\ 1 & j = k \end{cases}, \; j = 1(1)n-1, U_k^{'}(y_j) = 0, \; j = 1(1)n-1 \\ \text{and} \\ U_k(x_j) = 0, \quad j = 1(1)n \end{cases}$$

For $k = 1(1)n-1$

(3.2)
$$\begin{cases} V_k(y_j) = 0, \; j = 1(1)n-1, \quad V_k^{'}(y_j) = \begin{cases} 0 & j \neq k \\ & \text{for} \\ 1 & j = k \end{cases}, \; j = 1(1)n-1 \\ \text{and} \\ V_k(x_j) = 0, \quad j = 1(1)n \end{cases}$$

For $k = 1(1)n$

(3.3)
$$\begin{cases} W_k(y_j) = 0, \; j = 1(1)n-1, \; W_k^{'}(y_j) = 0, \; j = 1(1)n-1 \\ W_k(x_j) = \begin{cases} 0 & j \neq k \\ & \text{for} \\ 1 & j = k \end{cases}, \; j = 1(1)n \end{cases}$$

IV. TO PROVE THEOREM 2, WE NEED

Lemma 4.1

For $k = 1(1)n-1$ and $x \in (-\infty, \infty)$, we have,

$$\left|L_k(x)\right| = 0\left(\frac{2^n n! e^{\frac{v_1}{2}(x^2+y^2{}_k)}}{\sqrt{n} H_n^2(y_k)}\right), \quad v_1 > 1 \text{ and } k = 1(1)$$

where $L_k(x)$ is given by (1.3).

Proof.

From (2.9) at $y = y_k$ and using (1.3) and (2.2), we get

$$\left|L_k(x)\right| \le \frac{2^n (n-1)!}{H_n^2(y_k)} \sum_{i=0}^{n-1} \frac{1}{2^i i!} \left|H_i(x)\right| \left|H_i(y_k)\right|,$$

which on using (2.7) leads the lemma.

V. Estimation of the fundamental polynomials

Lemma 5.1:

For $k = 1(1)n-1$ and $x \in (-\infty, \infty)$

$$\sum_{k=1}^{n-1} e^{\beta y_k^2} \left| U_k(x) \right| = 0\left(\sqrt{n}\right) e^{vn^2}, v > \frac{3}{2} \text{ and } 0 \le \beta < 1,$$

where $U_k(x)$ is given by (1.6).

Proof.

From (1.6), we have

when $\left| x - y_k \right| < n^{1/2} \Big|$

$$\sum_{k=1}^{n-1} e^{\beta y_k^2} \left| U_k(x) \right| \le \sum_{k=1}^{n-1} \frac{e^{\beta y_k^2} L_k^2(x) \left| H_n(x) \right|}{\left| H_n(y_k) \right|}.$$

$$+ \sum_{k=1}^{n-1} \frac{2 e^{\beta y_k^2} \left| x_k \right| \left| x - y_k \right| L_k^2(x) H_n(x)}{\left| H_n(y_k) \right|}$$

(5.1) $$= I_1 + I_2$$

Using (2.7), (2.13) and lemma 4.1, we get

(5.2) $$I_1 = 0\left(\sqrt{n}\right) e^{\beta x^2}, v > \frac{3}{2}$$

Similarly, owing to (2.6), (2.7), (2.13) and lemma 4.1, we have

(5.3) $$I_2 = 0\left(\sqrt{n}\right) e^{\beta x^2}, v > \frac{3}{2}$$

On combining (5.2) and (5.3), we get the lemma.

When $\left| x - y_k \right| > n^{1/2}$, using (1.3), we have

$$\sum_{k=1}^{n-1} e^{\beta y_k^2} \left| U_k(x) \right| \le \sum_{k=1}^{n-1} \frac{e^{\beta y_k^2} \left| H_n(x) \right| L_k^2(x)}{\left| H_n(y_k) \right|}$$

$$+ \sum_{k=1}^{n-1} \frac{2 e^{\beta y_k^2} \left| y_k \right| \left| H_n(x) \right| \left| H_n'(x) \right| \left| L_k(x) \right|}{\left| H_n(y_k) \right| H_n'(y_k) \big| \ \big|}$$

$$= I_3 + I_4$$

From lemma 4.1,(2.7) and (2.13) we get

(5.4) $$I_3 = 0\left(\sqrt{n}\right)e^{vx^2}, \quad v > \frac{3}{2}$$

Similarly, using (2.6), (2.7), (2.13), lemma 4.1, (2.1) at $x = y_k$ and (2.2) we get

(5.5) $$I_4 = 0\left(\sqrt{n}\right)e^{vx^2}, \quad v > \frac{3}{2}$$

Owing to (5.4) and (5.5), we get the lemma.

Lemma 5.2

For $k = 1(1)n-1$ and $x \in (-\infty, \infty)$, we have

$$\sum_{k=1}^{n-1} e^{\beta y_k^2}\left|V_k(k)\right| \le \sum_{k=1}^{n-1} \frac{e^{\beta y_k^2}\left|H_n(x)\right|\left|H_n'(x)\right|\left|L_k(x)\right|}{\left|H_n(y_k)\right|\ \left|H_n''(y_k)\right|}$$

Using (2.1) at $x = y_k$, (2.2), (2.7), (2.13) and lemma 4.1, we get the required lemma.

Lemma 5.3

For $k = 1(1)n$ and $x \in (-\infty, \infty)$

$$\sum_{k=1}^{n} e^{\beta x_k^2}\left|W_k(x)\right| = 0\left(e^{vx^2}\right), v > \frac{3}{2} \text{ and } 0 \le \beta < 1.$$

Where $W_k(x)$ is given by (1.8).

Proof.

From (1.8), we have

$$\sum_{k=1}^{n} e^{\beta x_k^2}\left|W_k(x)\right| \le \sum_{k=1}^{n} \frac{e^{\beta x_k^2} H_n'^2(x)\left|1_k(x)\right|}{H_n'^2(x_k)}$$

Using (2.8), (2.10), (2.12) and (2.13), we get the lemma.

VI. IN THIS SECTION, WE MENTION CERTAIN THEOREMS OF G. FREUD AND L. SZILI REQUIRED IN THE PROOF OF THEOREM 2.

Theorem (G. Freud, Theorem 4[4] and theorem 1[3])

Let $f : R \to R$ be continuously differentiable. Further, let

$$\lim_{|x| \to +\infty} x^{2k}\rho(x) f(x) 0, \ k = 0,1,2,.... .$$

and

$$\lim_{|x| \to +\infty} x^{2k}\rho(x) f'(x) = 0,$$

then there exist polynomials $Q_n(x)$ of degree $\leq n$, such that

(6.1) $$\rho(x)\left|f(x)-Q_n(x)\right| = 0\left(\frac{1}{\sqrt{n}}\right)\omega\left(f';\frac{1}{\sqrt{n}}\right), \quad x \in R,$$

where ω stands for modulus of continuity defined by (1.12) and $\rho(x)$ the weight function.

Szili ([11] lemma 4, theorem 4) established the follow

(6.2) $$\rho(x)\left|Q_n^{(r)}(x)\right| = 0(1), r - 0,1: \ x \in R$$

VII. PROOF OF THE MAIN THEOREM 2.

(7.1) $$Q_n(x) = \sum_{k=1}^{n-1} Q_n(y_k)U_k(x) + \sum_{k=1}^{n-1} Q_n^{'}(y_k)V_k(x) + \sum_{k=1}^{n} Q_n(x_k)^{'} W_k(x)$$

From (7.1) and (1.11), we have

$$\left|R_n(x) - f(x)\right| \leq \left|R_n(f - Q_n)(x)\right| + \left|Q_n(x) - f(Y)\right|$$

$$e^{-vx^2}\left|R_n(x) - f(x)\right| \leq e^{-vx^2}\left|Q_n(x) - f(x)\right|$$
$$+ e^{-vx^2}\sum_{k=1}^{n-1} e^{-\beta y_k^2}\left|f(y_k) - Q_n(y_k)\right|\left|U_k(x)\right| e^{-\beta y_k^2}$$
$$+ e^{-vx^2}\sum_{k=1}^{n-1} \delta_k \left|V_k(x)\right|$$
$$+ e^{-vx^2}\sum_{k=1}^{n-1} e^{-\beta y_k^2}\left|Q_n^{'}(y_k)\right|\left|V_k(x)\right| e^{-\beta y_k^2}$$
$$+ e^{-vx^2}\sum_{k=1}^{n} e^{-\beta y_k^2}\left|f(x_k)\right| - Q_n\left|W_k(x)\right| e^{-\beta y_k^2}$$

Owing to (6.1), (6.2), (1.10) and lemmas 5.1-5.3, theorem follows

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