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Generalised Gaussian Quadrature over a triangle

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Abstract: - We introduce a Generalised Gaussian quadrature method for evaluation of the double integral I= $\iint_T f(x,y)dy\,dx$, where $f(x,y)$ is arbitrary function and T refers to the triangle region $\{(x,y)\ /\ 0 \le x \le a,\ 0 \le y \le a - x$, are derived using transformation of variables. new sampling points and its weight coefficients are calculated. We have then demonstrated the application of the derived quadrature rules by considering the evaluation of some typical integrals over the triangle region with various values of a.

Keywords: - *Finite element method, Generalised Gaussian quadrature , triangle region, extended numerical integration*

I. INTRODUCTION

The finite element method has become a powerful tool for the numerical solution of a wide range of engineering problem. In FEM various integrals are to be determined numerically in the evaluation of mass of a shell, center of mass and moments of inertia of a shell, fluid flow and mass flow across a surface, electric charge distributed over a surface, plate bending, plane strain, heat conduction over a plate, and similar problems in other areas of engineering which are very difficult to analyse using analytical techniques, These problems can be solved using the finite element method. The integrals in practical situations are not always simple but Gaussian quadrature that will evaluate these integrals exactly, the integration points have to be increased in order to improve the integration accuracy

From the literature review we may realize that several works in numerical integration using Gaussian quadrature over triangle region and have been carried out [1-10], and Generalized Gaussian Quadrature rules for system of arbitrary functions [12-13]

The method proposed here is termed as Generalized Gaussian rules, since the Generalized Gaussian quadrature nodes and weights for products of polynomial and logarithmic function given in [12] by Ma et al. are used in this paper

The paper is organized as follows. In Section III we will introduce the Generalized Gaussian quadrature formula over a triangle region of various values a. and In Section IV we compare the numerical results with some illustrative examples.

II. GENERALISED GAUSSIAN QUADRATURE OVER A TRIANGLE REGION

Generalized Gaussian quadrature rule for integrating function bounded by the triangle region
T = $\{(x,\ y)\ /\ 0 \le\ x \le\ a\ ,\ 0 \le y \le\ a - x\ \}$, with a=0.5, 1 ,2, 3 in the region T_1 , T_2, T_3 and T_4 respectively

III. FORMULATION OF INTEGRALS OVER A TRIANGLE REGION

The Numerical integration of an arbitrary function f over the triangle region is given by

$$I = \iint_T f(x,y)dx\,dy = \int_0^a \int_0^{a-x} f(x,y)dy\,dx = \int_0^a \int_0^{a-y} f(x,y)dx\,dy \quad (1)$$

The double integral over the triangle surface of equation(1) can be transformed to the standard square $\{(\xi,\eta)/\ 0 \le \xi \le 1, 0 \le \eta \le 1\}$ mathematical transformation is

$$x = a\,\xi \quad \text{and} \quad y = a\,\eta\,(1-\xi) \quad (2)$$

We have

$$I = \int_0^a \int_0^{a-x} f(x,y)dy\,dx = \int_0^1 \int_0^1 f\big(x(\xi,\eta), y(\xi,\eta)\big)\,J\,d\xi\,d\eta \quad (3)$$

Where $J(\xi,\eta)$ is the Jacobians of the transformation

$$J(\xi\,,\eta)=\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{vmatrix}=a^2(1-\,\xi)$$

From equation(3) , we can write as

$$I=\int_0^1\int_0^1 f(a\,\xi\,,a\,\eta\,(1-\,\xi))\,a^2(1-\,\xi)\,d\xi\,d\eta$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n} a^2(1-\,\xi)\;w_i\,w_j\,f\big(x(\xi_i\,,\eta_j)\,,y(\xi_i\,,\eta_j)\big) \qquad (4)$$

Where $\xi_{i\,,}\,\eta_j$ are sampling points and corresponding to its weight coefficients w_i, w_j . We can rewrite equation (4) as

$$\mathrm{I}=\sum_{k}^{N=n\times n} W_k\,f(x_k,y_k) \qquad (5)$$

Where $W_k=a^2(1-\,\xi)\;\;w_i\,w_j$ (5a)

$x_k=\;a\,\xi$ and $y_k=a\,\eta\,(1-\,\xi)$, (5b)

if $k,i,j=1,2,3,----$

we find out new sampling points x_k , y_k and weights coefficients W_k of various order N =5, 10, 15, 20 by using equations(5a) and (5b)

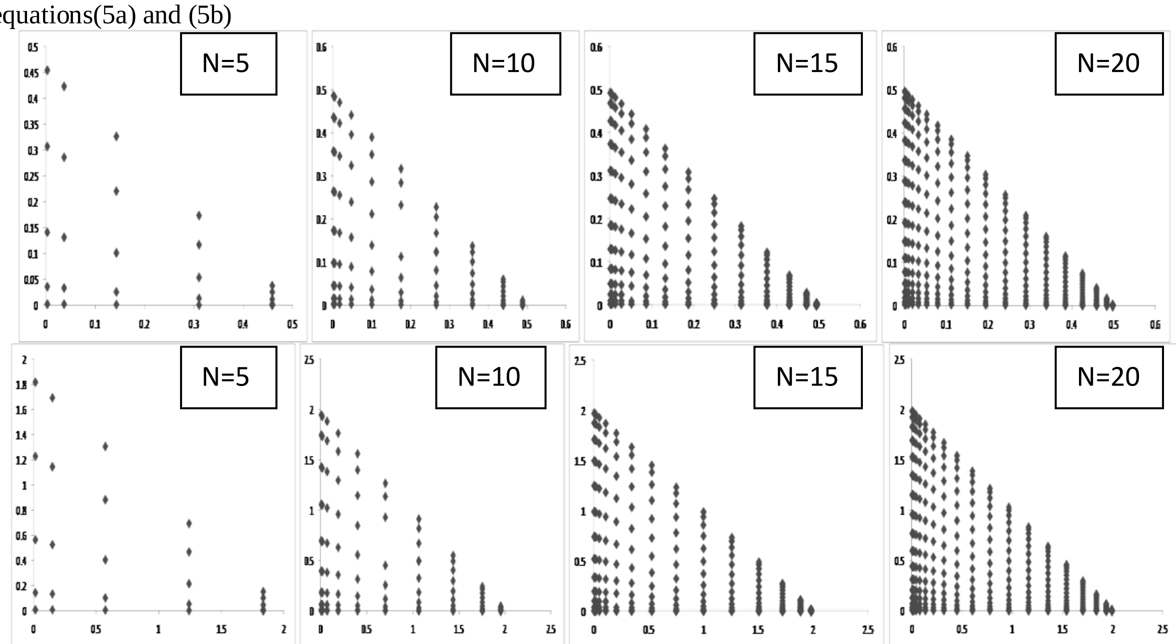

Fig. 1 Sampling points (x_k, y_k) values for the region T_1 and T_3

Region T_1 for N=5			Region T_2 for N=5		
x_k	y_k	W_k	x_k	y_k	W_k
0.0028261141	0.0028101402	0.0001101175	0.0056522282	0.0056202805	0.0004404701
0.0367151858	0.0026185914	0.0006372372	0.0734303717	0.0052371829	0.0025489489
0.1424787022	0.0020207919	0.0010899659	0.2849574044	0.0040415839	0.0043598639
0.3097411320	0.0010753865	0.0007012056	0.6194822640	0.0021507730	0.0028048225
0.4578790415	0.0002380772	0.0000923418	0.9157580830	0.0004761545	0.0003693672
0.0028261141	0.0365076632	0.0006838508	0.0056522282	0.0730153265	0.0027354034
0.0367151858	0.0340191761	0.0039573646	0.0734303717	0.0680383522	0.0158294584
0.1424787022	0.0262529218	0.0067688963	0.2849574044	0.0525058436	0.0270755855
0.3097411320	0.0139707794	0.0043546205	0.6194822640	0.0279415588	0.0174184820
0.4578790415	0.0030929576	0.0005734602	0.9157580830	0.0061859152	0.0022938409
0.0028261141	0.1416733800	0.0015157212	0.0056522282	0.2833467601	0.0060628850
0.0367151858	0.1320164381	0.0087713009	0.0734303717	0.2640328763	0.0350852038
0.1424787022	0.1018783410	0.0150029207	0.2849574044	0.2037566821	0.0600116828
0.3097411329	0.0542156731	0.0096517988	0.6194822640	0.1084313463	0.0386071952
0.4578790415	0.0120026790	0.0012710459	0.9157580830	0.0240053580	0.0050841838
0.0028261141	0.3079904044	0.0018323515	0.0056522282	0.6159808089	0.0073294061
0.0367151858	0.2869967255	0.0106036032	0.0734303717	0.5739934511	0.0424144130
0.1424787022	0.2214781030	0.0181369923	0.2849574044	0.4429562060	0.0725479695
0.3097411320	0.1178619942	0.0116680348	0.6194822640	0.2357239885	0.0466721394
0.4578790415	0.0260931867	0.0015365642	0.9157580830	0.0521863734	0.0061462568
0.0028261141	0.4552910046	0.0010899547	0.0056522282	0.9105820093	0.0043598188
0.0367151858	0.4242568132	0.0063074399	0.0734303717	0.8485136265	0.0252297599
0.1424787022	0.3274030182	0.0107885959	0.2849574044	0.6548060365	0.0431543838
0.3097411320	0.1742310961	0.0069406057	0.6194822640	0.3484621923	0.0277624230
0.4578790415	0.0385726082	0.0009140087	0.9157580830	0.0771452164	0.0036560351

Region T_3 for N=5			Region T_4 for N=5		
x_k	y_k	W_k	x_k	y_k	W_k
0.0113044564	0.0112405610	0.0017618805	0.0169566846	0.0168608415	0.0039642312
0.1468607434	0.0104743659	0.0101957958	0.2202911152	0.0157115489	0.0229405407
0.5699148089	0.0080831678	0.0174394559	0.8548722133	0.0121247517	0.0392387759
1.2389645281	0.0043015461	0.0112192901	1.8584467922	0.0064523192	0.0252434028
1.8315161660	0.0009523090	0.0014774689	2.7472742490	0.0014284636	0.0033243052
0.0113044564	0.1460306530	0.0109416136	0.0169566846	0.2190459794	0.0246186307
0.1468607434	0.1360767044	0.0633178337	0.2202911152	0.2041150567	0.1424651259
0.5699148089	0.1050116872	0.1083023421	0.8548722133	0.1575175308	0.2436802699
1.2389645281	0.0558831176	0.0696739281	1.8584467922	0.0838246764	0.1567663382
1.8315161660	0.0123718305	0.0091753637	2.7472742490	0.0185577458	0.0206445684
0.0113044564	0.5666935203	0.0242515401	0.0169566846	0.8500402805	0.0545659652
0.1468607434	0.5280657526	0.1403408154	0.2202911152	0.7920986289	0.3157668347
0.5699148089	0.4075133642	0.2400467312	0.8548722133	0.6112700463	0.5401051452
1.2389645281	0.2168626927	0.1544287811	1.8584467922	0.3252940391	0.3474647575
1.8315161660	0.0480107160	0.0203367354	2.7472742490	0.0720160740	0.0457576546
0.0113044564	1.2319616179	0.0293176247	0.0169566846	1.8479424268	0.0659646556
0.1468607434	1.1479869022	0.1696576521	0.2202911152	1.7219803534	0.3817297173
0.5699148089	0.8859124120	0.2901918782	0.8548722133	1.3288686180	0.6529317260
1.2389645281	0.4714479771	0.1866885577	1.8584467922	0.7071719657	0.4200492550
1.8315161660	0.1043727469	0.0245850273	2.7472742490	0.1565591204	0.0553163114
0.0113044564	1.8211640186	0.0174392755	0.0169566846	2.7317460280	0.0392383699
0.1468607434	1.6970272530	0.1009190399	0.2202911152	2.5455408796	0.2270678399
0.5699148089	1.3096120731	0.1726175352	0.8548722133	1.9644181096	0.3883894543
1.2389645281	0.6969243847	0.1110496920	1.8584467922	1.0453865771	0.2498618070
1.8315161660	0.1542904328	0.0146241405	2.7472742490	0.2314356492	0.0329043163

Table 1 sampling points and weights coefficient over the region T for N = 5

IV. NUMERICAL RESULT

Exact value	Order	Computed value
1) $\int_0^1 \int_0^{1-x} e^{-y^2} \cos(xy)dydx = 0.4284998849$	N=5 N=10 N=15 N=20	0.4284990470 0.4295549333 0.4284994331 0.4284999082
2) $\int_0^1 \int_0^{1-y} \sqrt{x+y}dxdy = 0.4000000000$	N=5 N=10 N=15 N=20	0.4000000821 0.4010371532 0.3998678727 0.3999999865
3) $\int_0^{0.5} \int_0^{0.5-x} \frac{x^4+y^4}{1+x^2y}dydx = 0.001036548993$	N=5 N=10 N=15 N=20	0.0010367635 0.0010539658 0.0010365468 0.0010365492
4) $\int_0^{0.5} \int_0^{0.5-x} \frac{1}{\sqrt{1-x-y}}dydx = 0.1548220315$	N=5 N=10 N=15 N=20	0.1548220216 0.1550437840 0.1548156532 0.1548220361
5) $\int_0^2 \int_0^{2-x} \frac{1}{\sqrt{(1+x)^2+y^2}}dydx = 1.120666914$	N=5 N=10 N=15 N=20	1.1206837170 1.1144932656 1.1212498595 1.1206668174
6) $\int_0^2 \int_0^{2-y} \frac{e^{-x}}{\sqrt{1+xy}}dxdy = 1.014219466$	N=5 N=10 N=15 N=20	1.0142448777 1.0134605001 1.0149180621 1.0142193709

7) $\int_0^3 \int_0^{3-x} \frac{x^2+y^2}{\sqrt{1+x+y}} dydx = 7.390476190$	N=5 N=10 N=15 N=20	7.3904818590 7.4061649340 7.3895392081 7.3904788091
8) $\int_0^3 \int_0^{3-x} \sin(x+y+1) dydx = 0.362657383$	N=5 N=10 N=15 N=20	0.3625743827 0.3648534035 0.3621653639 0.3626564613

V. CONCLUSIONS

In this paper we derived Generalised Gaussian quadrature method for calculating integral over a triangle region $\{(x, y) \ / \ 0 \le x \le a, 0 \le y \le a - x\}$ with a = 0.5, 1, 2, 3. New sampling points and its weight coefficients are calculated of various order N = 5, 10, 15, 20. We have then evaluate the typical integrals Governed by the proposed method. The results obtained are in excellent agreement with the exact value.

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