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The Solve of Laplace Equation with Nonlocal and Derivative Boundary Conditions by Using Non Polynomial Spline

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Abstract: **-** In this paper we consider a non polynomial spline function where it to depend on parameter τ, such that it interpolated *u(x)* in points of grid. By using this function we solve the Laplace equation with nonlocal and derivative boundary conditions. The method is applied in this paper is a implicit method. We know superiority of implicit methods is stability of them, because most of them are unconditionally stable.

Key Words: **-** *Non polynomial function, Cubic spline, Laplace equation, Implicit method, Nonlocal and derivative boundary conditions.*

I. INTRODUCTION

The Laplace equation is a elliptic equation that occur in many branches of applied mathematics. In this paper we consider the Laplace equation $\frac{\partial^2 u}{\partial_{x^2}} + \frac{\partial^2 u}{\partial_{y^2}} = 0$ with the following initial-boundary conditions

$$u_x(1, y) = g(y) \qquad 0 < y < Y \qquad (1)$$

$$\int_0^1 u(x, y)dx = m(y) \qquad 0 < y < Y \qquad (2)$$

$$u(x, 0) = f(x) \qquad 0 < y < Y \qquad (3)$$

Where $f(x), g(y)$ and $m(y)$ are known, while the function $u(x, y)$ is to be determined, the boundary value problem arises in a large variety of applications in engineering, physics and other science. Study in this filed is very important. You can see some methods for solve of these problems [3, 4, 5]. In many problems boundary conditions are as integral equations. These problems are a kind of nonlocal problems and have important applications in other branches of pure and applied science [6, 7, 8, 9]. In this paper we want to solve a type of nonlocal problem by using a non polynomial spline function. Rashidinia and Jalilian used a non polynomial spline function to smooth the approximate solution of the second order boundary value problems [2]. In addition they used quintic non polynomial spline functions to develop numerical methods for approximation to the solution of a system of fourth order boundary value problems associated with obstacle, unilateral and contact problems [1].

II. THE PRESENTATION NON POLYNOMIAL SPLINE FUNCTIONS

Note that grid points P are given by $x_i,\ i = 0,1, \dots, n\ as\ follows$

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Where $x_i = a + ih,\ i = 0,1, \dots, n\ and\ h = (b - a)/n.$

Consider non polynomial function $Sp(x) \in C^2[a, b]$ such that depend on parameter τ and interpolate $u(x)$ in grid points. When $\tau \to 0$ this function is converted to ordinary cubic spline in [a,b].

For all subinterval $[x_i, x_{i+1}],\ i = 0,1, \dots, n - 1$ non polynomial function $Sp(x)$ is defined as follow

$$Sp(x) = a_i + b_i(x - x_i) + c_i \sin\tau(x - x_i) + d_i \cos\tau(x - x_i) \qquad i = 0,1, \dots, n - 1 \qquad (4)$$

Where a_i, b_i, c_i and d_i are constant numbers, and τ is a ordinary parameter. u_i is considered as a approximation $u(x_i)$ where is got by spline function that pass through points (x_i, u_i) and (x_{i+1}, u_{i+1}).
Now we obtain necessary conditions to presentation coefficients in E.q. (4). First must function $Sp(x)$ satisfies in interpolation conditions in points x_i and x_{i+1}, namely $Sp(x_i) = u_i$ and $Sp(x_{i+1}) = u_{i+1}$. We know that the first derivative of function Sp is continued in common points x_i and x_{i+1} with M_i and M_{i+1} respectively. Namely $S''p(x_i) = M_i$ and $S''p(x_{i+1}) = M_{i+1}$. It follows from E.q. (4) that

$$a_i = u_i + \frac{M_i}{\tau^2} \qquad d_i = -\frac{M_i}{\tau^2} \qquad (5)$$

Since $s(x_{i+1}) = u_{i+1}$ and $s''(x_{i+1}) = M_{i+1}$ we obtain

$$b_i = \frac{u_{i+1}-u_i}{h} + \frac{M_{i+1}-M_i}{\tau\theta} \qquad c_i = \frac{M_i\cos\theta - M_{i+1}}{\tau^2 \sin\theta}\ , \quad i = 1,2,\ldots,n-1 \qquad (6)$$

where $\theta = \tau h$. Noting that $S'^{p_{i-1}}(x_i)$, hence we have

$$b_{i-1} + \tau c_{i-1}\cos\tau(x - x_{i-1}) - \tau d_{i-1}\sin\tau(x - x_{i-1}) = b_i \tau c_i \cos\tau(x - x_i) - \tau d_i \sin\tau(x - x_i)$$
(7)

substituting E.q.s (5) and (6) into E.q. (7) by simplifying implies that

$$\frac{1}{\theta^2}[\theta\csc\theta - 1]M_{i+1} + \frac{2}{\theta^2}[1 - \theta\cot\theta]M_i + \frac{1}{\theta^2}[\theta\csc\theta - 1]M_{i-1} = \frac{1}{h^2}[u_{i+1} - 2u_i + u_{i-1}]$$

Finally, by letting $\alpha = \frac{1}{\theta^2}(\theta\csc\theta - 1)$ and $\beta = \frac{1}{\theta^2}(1 - \theta\cot\theta)$ we obtain

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) \qquad (8)$$

if $\tau \to 0$ then $\theta \to 0$, therefore

$$\alpha = \lim_{\theta\to 0}\frac{\theta\csc\theta - 1}{\theta^2} = \frac{1}{6} \qquad \beta = \lim_{\theta\to 0}\frac{1 - \theta\coth\theta}{\theta^2} = \frac{1}{3}$$

Thus $(\alpha, \beta) \to (\frac{1}{6}, \frac{1}{3})$, and (8) changed into ordinary cubic spline

$$\frac{h^2}{6}(M_{i+1} + 4M_i + M_{i-1}) = u_{i+1} - 2u_i + u_{i-1}$$

III. FINIT DIFFERENCE SCHEME FOR LAPLACE EQUATION

Cover the domain $[0,1] \times [0,\infty]$ by $\Omega_h \times \Omega_k$, where $\Omega_h = \{x_i | x_i = ih, i = 0,1,\ldots N\}$ and $\Omega_k = \{y_i | y_i = jh,\ j = 0,1,2,\ldots\}$. In which N is a positive integer and h is step size in space.
The notations u_i^j and M_i^j are used for the finite difference approximations of $u(x_i, y_j)$ and $s''(x_i, y_j)$, respectively. We approximate the derivative u_{yy} in Laplace equation by

$$(u_{yy})_i^j = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{h^2}$$

And the derivative of space u_{xx} by non polynomial cubic spline function. Therefore the finite difference scheme for Laplace equation is as follows:

$$\frac{1}{h^2}(u_i^{j+1} - 2u_i^j + u_i^{j-1}) = M_i^j \qquad (9)$$

from (8) and (9) we have

$$\alpha u_{i+1}^{j+1} + 2\beta u_i^{j+1} + \alpha u_{i-1}^{j+1} = (2\alpha - r^2)u_{i+1}^j + (4\beta + 2r^2)u_i^j + (2\alpha - r^2)u_{i-1}^j$$
$$-[\alpha u_{i+1}^{j-1} + 2\beta u_i^{j-1} + \alpha u_{i-1}^{j-1}] \qquad (10)$$

and $r = k/h$. previously parameters α and β are defined.
Differentiating E.q. (2) with respect to y and then using Laplace equation, we have condition:

$$u_x(0, y) = g(y) + m''(y) \qquad 0 < y < Y \qquad (11)$$

We approximate u_x at $x=0$ and $x=N$ by a central difference formula. Then the boundary conditions can be represented by

$$\frac{u_1^j - u_{-1}^j}{2h} = u_0^j \qquad \frac{u_{N+1}^j - u_{N-1}^j}{2h} = u_N^j$$

u_{-1}^j and u_{N+1}^j are unknown that they obtain by boundary conditions E.q. (1) and (11). By using these equations, linear system of equations is as follows

$$\begin{pmatrix} 2\beta - 2\alpha h & 2\alpha & & & \\ \alpha & 2\beta & \alpha & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & 2\beta & \alpha \\ & & & 2\alpha & 2\beta + 2\alpha h \end{pmatrix} \begin{pmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ u_{N-1}^{j+1} \\ u_N^{j+1} \end{pmatrix} =$$

$$\begin{pmatrix} a & 2(2\alpha - r^2) & & & \\ 2\alpha - r^2 & 4\beta + 2r^2 & 2\alpha - r^2 & & \\ \ddots & \ddots & \ddots & & \\ & 2\alpha - r^2 & 4\beta + 2r^2 & 2\alpha - r^2 \\ & & 2(2\alpha - r^2) & b \end{pmatrix} \begin{pmatrix} u_0^j \\ u_1^j \\ \vdots \\ u_{N-1}^j \\ u_N^j \end{pmatrix} -$$

$$\begin{pmatrix} 2\beta - 2\alpha h & 2\alpha & & & \\ \alpha & 2\beta & \alpha & & \\ \ddots & \ddots & \ddots & & \\ & & \alpha & 2\beta & \alpha \\ & & & 2\alpha & 2\beta + 2\alpha h \end{pmatrix} \begin{pmatrix} u_0^{j-1} \\ u_1^{j-1} \\ \vdots \\ u_{N-1}^{j-1} \\ u_N^{j-1} \end{pmatrix}$$

where $a = (4\beta + 2r^2) - 2(2\alpha - r^2)h$, $b = (4\beta + 2r^2) + 2(2\alpha - r^2)h$, giving

$$AU^{j+1} = BU^j + CU^{j-1} \qquad (12)$$

where A, B and C are as displayed, U^{j+1} is a vector of unknown values and U^{j} and U^{j-1} are vector of known values. The finite difference approximation of a Laplace equation needs three space-levels. For solve the first set of equation for $u_{i,2}$ it is necessary to calculate a solution along the first space-level by some other method, it being assumed that the initial data along $y=0$ are known.

We get different methods for solve Laplace equation by suitable choosing parameters α and β. With condition $0 \leq \alpha \leq \beta$ the matrix A is a diagonally dominate matrix. Therefore from E.q. (12) we get

$$U^{j+1} = A^{-1}BU^{j} + A^{-1}CU^{j-1}$$

Error, stability and compatibility conditions are presented in the next section.

IV. STABILITY AND COMPATIBILITY OF METHOD AND ERROR

We let the solution of E.q. (10) in point (x_i, y_j) is as follows

$$u_i^j = \xi^j e^{i\sqrt{-1}\theta} \qquad (13)$$

θ is real and ξ is complex. By using (10) and (13) we get

$$2(\alpha\cos\theta + \beta)\xi^2 - 2[(2\alpha - r^2)\cos\theta + (2\beta + r^2)]\xi + 2[\alpha\cos\theta + \beta] = 0$$

by putting $A = 2(\alpha\cos\theta + \beta)$ and $B = 2[(2\alpha - r^2)\cos\theta + (2\beta + r^2)]$ we obtain

$$A\xi^2 - B\xi + A = 0 \qquad (14)$$

we assume that the roots of E.q. (14) are ξ_1 and ξ_2 , therefore $\xi_1.\xi_2 = 1$. We have two states:

1) $|\xi_1| \neq |\xi_2| \neq 1$ thus $|\xi_1| = 1/|\xi_2|$ so $|\xi_1| < 1$, $|\xi_1| > 1$ (or $|\xi_1| > 1$, $|\xi_2| < 1$) therefore in this case our scheme is not stable.

2) $|\xi_1| = |\xi_2| = 1$, in this case $\cos\theta = 0$ so $\theta = 0, 2\pi, \ldots$. We know $\theta = 0$ is not acceptable, hence we admit $\theta = 2\pi$.

For study of error and compatibility, we let $D_x = \frac{\partial}{\partial_x}$ and $D_y = \frac{\partial}{\partial_y}$. Then by Taylors expansion of terms in E.q. (10) we obtain

$$2(\alpha+\beta)k^2D_y^2 + \frac{1}{3!}(\alpha+\beta)k^4D_y^4 + \frac{1}{180}2(\alpha+\beta-1)k^6D_y^6 + \alpha h^2k^2D_x^2D_t^2 + r^2h^2D_x^2 + \frac{1}{4!}(2r^2 - 4\alpha)h^4D_x^4 + \ldots$$

Thus by suitable choosing of parameters α and β we obtain different error. In addition our difference scheme is compatible for Laplace equation.

V. CONCLUSIONS

Our purpose in this article is solving partial differential equation. We presented an explicit method for solving this equation, and we considered necessary conditions for stability of this method.

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