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Some Fixed Point and Common Fixed Point Theorems in 2-Banach Spaces

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Abstract: In the present paper we prove some fixed point and common fixed point theorems in 2-Banach spaces for new rational expression. Which generalize the well known results.
AMS: 47H10, 54H25.

Keywords*: Banach Space, 2-Banach Spaces, Fixed point, Common Fixed point.*

I. INTRODUCTION

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis. It is well known that the differential and integral equations that arise in physical problems are generally non-linear, therefore the fixed point methods specially Banach's contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods. Recently Verma [9] described about the application of Banach's contraction principle [2]. Ghalar [5] introduced the concept of 2-Banach spaces. Recently Badshah and Gupta [3], Yadava, Rajput and Bhardwaj [10], Yadava, Rajput, Choudhary and Bhardwaj [11] also worked for Banach and 2-Banch spaces for non contraction mappings. In present paper we prove some fixed point and common fixed point theorems for non-contraction mappings, in 2-Banach spaces motivated by above, before starting the main result first we write some definitions.

Definition (1.2A), 2-Banach Spaces:

In a paper Gahler [5] define a linear 2-normed space to be pair $(L,\|.,.\|)$ where L is a linear space and $\|.,.\|$ is non negative , real valued function defined on L such that $a,b,c \in L$

(i) $\|a,b\| = 0$ if and only if a and b are linearly dependent

(ii) $\|a,b\| = \|b,a\|$

(iii) $\|a,\beta b\| = |\beta| \, \|a,b\|$, β is real

(iv) $\|a,b+c\| \le \|a,b\| + \|a,c\|$

Hence $\|.,.\|$ is called a 2-norm.

Definition (1.2B):

A sequence $\{x_n\}$ in a linear 2-normed space L, is called a convergent sequence if there is, $x \in L$, such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in L$.

Definition (1.2C):

A sequence $\{x_n\}$ in a linear 2-normed space L , is called a Cauchy sequence if there exists $y,z \in L$, such that y and z are linearly independent and

$$\lim_{m,n\to\infty}\|x_m - x_n, y\| = 0$$

Definition (1.2D):
A linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach spaces.

II. MAIN RESULTS

Theorem 1.1:
Let T be a mapping of a 2-Banach spaces into itself. If T satisfies the following conditions:

$$T^2 = I \text{ , where } I \text{ is identity mapping} \tag{1.1}$$

$$\|Tx-Ty,a\| \ge \alpha\frac{\|x-Tx,a\|\|y-Ty,a\|}{\|x-y,a\|} + \beta\frac{\|y-Ty,a\|\|y-Tx,a\|\|x-Ty,a\| + \|x-y,a\|^3}{\|x-y,a\|^2}$$
$$+\gamma\left[\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}\right] + \delta\left[\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}\right] + \eta\|x-y,a\| \tag{1.2}$$

Where $x \neq y$, $a > 0$ is real with $8\alpha + 10\beta + 4\gamma + 2\delta + 3\eta > 4$. Then T has unique fixed point.

Proof: Suppose x is any point in 2-Banach space X.

Taking $y = \frac{1}{2}(T+I)x \quad , \quad z = T(y)$

$$\|z-x,a\| = \|Ty - T^2x,a\| = \|Ty - T(Tx),a\|$$

$$\ge \alpha\frac{\|y-Ty,a\|\|Tx-T(Tx),a\|}{\|y-Tx,a\|} + \beta\frac{\|Tx-T(Tx),a\|\|Tx-Ty,a\|\|y-T(Tx),a\| + \|y-Tx,a\|^3}{\|y-Tx,a\|^2}$$

$$+\gamma\left[\frac{\|y-Ty,a\|+\|Tx-T(Tx),a\|}{2}\right] + \delta\left[\frac{\|y-T(Tx),a\|+\|Tx-Ty,a\|}{2}\right] + \eta\|y-Tx,a\|$$

$$\ge \alpha\frac{\|y-Ty,a\|\|Tx-x,a\|}{\frac{1}{2}\|x-Tx,a\|} + \beta\frac{\|Tx-x,a\|[\|Tx-y,a\|+\|y-Ty,a\|]\|y-x,a\| + \|y-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}$$

$$+\gamma\left[\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}\right] + \delta\left[\frac{\|y-x,a\|+\|Tx-y,a\|+\|y-Ty,a\|}{2}\right] + \eta\|y-Tx,a\|$$

$$\ge 2\alpha\|y-Ty,a\| + \beta\frac{\|Tx-x,a\|\left[\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|\right]\frac{1}{2}\|x-Tx,a\| + \frac{1}{8}\|x-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}$$

$$+\gamma\left[\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}\right] + \delta\left[\frac{\frac{1}{2}\|x-Tx,a\|+\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|}{2}\right] + \frac{\eta}{2}\|x-Tx,a\|$$

$$\ge 2\alpha\|y-Ty,a\| + \frac{\beta}{2}\left\{4\left[\frac{1}{2}\|x-Tx,a\|+\|y-Ty\|\right] + \frac{\|x-Tx,a\|^3}{\|x-Tx,a\|^2}\right\}$$

$$+\gamma\left[\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}\right] + \delta\left[\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}\right] + \frac{\eta}{2}\|x-Tx,a\|$$

$$\geq 2\alpha\|y-Ty,a\|+\frac{\beta}{2}\left[2\|x-Tx,a\|+4\|y-Ty,a\|+\|x-Tx,a\|\right]$$

$$+\frac{\gamma}{2}\left[\|y-Ty,a\|+\|Tx-x,a\|\right]+\frac{\delta}{2}\left[\|x-Tx,a\|+\|y-Ty,a\|\right]+\frac{\eta}{2}\|x-Tx,a\|$$

$$\geq\|x-Tx,a\|\left(\frac{3\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}\right)+\left(2\alpha+2\beta+\frac{\gamma}{2}+\frac{\delta}{2}\right)\|y-Ty,a\|$$

$$\geq\frac{1}{2}\|x-Tx,a\|(3\beta+\gamma+\delta+\eta)+\frac{1}{2}\|y-Ty,a\|(4\alpha+4\beta+\gamma+\delta)$$

Now for

$$\|u-x,a\|=\|2y-z-x,a\|=\|Tx-Ty,a\|$$

$$\geq\alpha\frac{\|x-Tx,a\|\|y-Ty,a\|}{\|x-y,a\|}+\beta\frac{\|y-Ty,a\|\|y-Tx,a\|\|x-Ty,a\|+\|x-y,a\|^{3}}{\|x-y,a\|^{2}}$$

$$+\gamma\left[\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}\right]+\delta\left[\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}\right]+\eta\|x-y,a\|$$

$$\geq\alpha\frac{\|x-Tx,a\|\|y-Ty,a\|}{\frac{1}{2}\|x-Tx,a\|}+\beta\frac{\|y-Ty,a\|\frac{1}{2}\|x-Tx,a\|\left[\frac{1}{2}\|x-Tx,a\|\right]+\frac{1}{8}\|x-Tx,a\|^{3}}{\frac{1}{4}\|x-Tx,a\|^{2}}$$

$$+\gamma\left[\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}\right]+\delta\left[\frac{\frac{1}{2}\|x-Tx,a\|+\frac{1}{2}\|x-Tx,a\|}{2}\right]+\frac{\eta}{2}\|x-Tx,a\|$$

$$\geq 2\alpha\|y-Ty,a\|+\beta\|y-Ty,a\|+\frac{\beta}{2}\|x-Tx,a\|+\gamma\left[\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}\right]$$

$$+\frac{\delta}{2}\|x-Tx,a\|+\frac{\eta}{2}\|x-Tx,a\|$$

$$\geq\|x-Tx,a\|\left(\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}+\frac{\eta}{2}\right)+\|y-Ty,a\|\left(2\alpha+\beta+\frac{\gamma}{2}\right)$$

$$\geq\frac{1}{2}\|x-Tx,a\|(\beta+\gamma+\delta+\eta)+\frac{1}{2}\|y-Ty,a\|(4\alpha+2\beta+\gamma)$$

Now

$$\|z-u,a\|=\|z-x,a\|+\|x-u,a\|$$

$$\geq\frac{1}{2}\|x-Tx,a\|(3\beta+\gamma+\delta+\eta)+\frac{1}{2}\|y-Ty,a\|(4\alpha+4\beta+\gamma+\delta)+\frac{1}{2}\|x-Tx,a\|(\beta+\gamma+\delta+\eta)$$

$$+\frac{1}{2}\|y-Ty,a\|(4\alpha+2\beta+\gamma)$$

$$\geq\frac{1}{2}\|x-Tx,a\|(3\beta+\gamma+\delta+\eta+\beta+\gamma+\delta+\eta)+\frac{1}{2}\|y-Ty,a\|(4\alpha+4\beta+\gamma+\delta+4\alpha+2\beta+\gamma)$$

$$\geq\frac{1}{2}\|x-Tx,a\|(4\beta+2\gamma+2\delta+2\eta)+\frac{1}{2}\|y-Ty,a\|(8\alpha+6\beta+2\gamma+\delta) \tag{1.3}$$

On the other hand

$$\|z-u,a\| = \|T(y)-(2y-z),a\|$$
$$= \|T(y)-2y+T(y),a\|$$
$$= 2\|Ty-y,a\| \qquad (1.4)$$

So $2\|Ty-y,a\| \geq \frac{1}{2}\|x-Tx,a\|(4\beta+2\gamma+2\delta+2\eta)+\frac{1}{2}\|y-Ty,a\|(8\alpha+6\beta+2\gamma+\eta)$

$$\Rightarrow [4-(8\alpha+6\beta+2\gamma+\eta)]\|y-Ty,a\| \geq (4\beta+2\gamma+2\delta+2\eta)\|x-Tx,a\|$$

$$\Rightarrow \|x-Tx,a\| \leq \frac{4-(8\alpha+6\beta+2\gamma+\eta)}{4\beta+2\gamma+2\delta+2\eta}\|y-Ty,a\|$$

$\Rightarrow \|x-Tx,a\| \leq k\|y-Ty,a\|$ as $(8\alpha+10\beta+4\gamma+2\delta+3\eta>4)$

Where $k = \frac{4-(8\alpha+6\beta+2\gamma+\eta)}{4\beta+2\gamma+2\delta+2\eta} < 1$

Let $R = \frac{1}{2}(T+I)$, then

$$\|R^2(x)-R(x),a\| = \|R(R(x))-R(x),a\|$$
$$= \|R(y)-y,a\| = \frac{1}{2}\|y-Ty,a\|$$
$$< \frac{k}{2}\|x-Tx,a\|$$

By the definition of R we claim that $\{R^n(x)\}$ is a Cauchy sequence in X , $\{R^n(x)\}$ is converges to so element x_0 in X . So $\lim_{n\to\infty}\{R^n(x)\} = x_0$. So $\{R(x_0)\} = x_0$. Hence $T(x_0) = x_0$

So x_0 is a fixed point of T .

Uniqueness:

If possible let $y_0 \neq x_0$ is another fixed point of T .Then

$$\|x_0-y_0,a\| = \|Tx_0-Ty_0,a\|$$

$$\geq \alpha\frac{\|x_0-Tx_0,a\|\|y_0-Ty_0,a\|}{\|x_0-y_0,a\|} + \beta\frac{\|y_0-Ty_0,a\|\|y_0-Tx_0,a\|\|x_0-Ty_0,a\|+\|x_0-y_0,a\|^3}{\|x_0-y_0,a\|^2}$$

$$+\gamma\left[\frac{\|x_0-Tx_0,a\|\|y_0-Ty_0,a\|}{2}\right] + \delta\left[\frac{\|x_0-Ty_0,a\|+\|y_0-Tx_0,a\|}{2}\right] + \eta\|x_0-y_0,a\|$$

$$\geq \beta\|x_0-y_0,a\| + \delta\|x_0-y_0,a\| + \eta\|x_0-y_0,a\|$$

$$\geq (\beta+\delta+\eta)\|x_0-y_0,a\|$$

Which is contradiction so $x_0 = y_0$ .Hence fixed point in unique.

Theorem 2.1:

Let T and G be two expansion mappings of a 2-Banach space X into itself. T and G satisfy the following conditions:

(2.1) T and G commute

(2.1) $T^2 = I$ and $G^2 = I$, where I is identity mapping.

(2.3)

$$\|Tx-Ty,a\| \geq \alpha \frac{\|Gx-Tx,a\|\|Gy-Ty,a\|}{\|Gx-Gy,a\|} + \beta \frac{\|Gy-Ty,a\|\|Gy-Tx,a\|\|Gx-Ty,a\| + \|Gx-Gy,a\|^3}{\|Gx-Gy,a\|^2}$$

$$+\gamma\left[\frac{\|Gx-Tx,a\| + \|Gy-Ty,a\|}{2}\right] + \delta\left[\frac{\|Gx,Ty,a\| + \|Gy-Tx,a\|}{2}\right] + \eta\|Gx-Gy,a\|$$

For every $x, y \in X, \alpha, \beta, \gamma, \delta, \eta \in [0.1]$ with $x \neq y$ and $\|Gx-Gy\| \neq 0$ and $\beta+\delta+\eta > 1$.

Then there exists a unique common fixed of T and G such that $T(x_0) = x_0$ and $G(x_0) = x_0$.

Proof: -

Suppose x is point in 2-Banach space X it is clear that $(TG)^2 = I$

$$\|TG.G(x) - TG.G(y), a\| \geq \alpha \frac{\|G(G^2x) - T(G^2x), a\|\|G(G^2y) - T(G^2y), a\|}{\|G(G^2x) - G(G^2y), a\|}$$

$$+\beta \frac{\|G(G^2y) - T(G^2y), a\|\|G(G^2y) - T(G^2x), a\|\|G(G^2x) - T(G^2y), a\| + \|G(G^2x) - G(G^2y), a\|^3}{\|G(G^2x) - G(G^2y), a\|^2}$$

$$+\gamma\left[\frac{\|G(G^2x) - T(G^2x), a\| + \|G(G^2y) - T(G^2y), a\|}{2}\right] + \delta\left[\frac{\|G(G^2x) - T(G^2y), a\| + \|G(G^2y) - T(G^2x), a\|}{2}\right]$$

$$+\eta\|G(G^2x) - G(G^2y), a\|$$

$$\geq \alpha \frac{\|Gx - TG(Gx), a\|\|Gy - TG(Gy), a\|}{\|Gx-Gy,a\|^2} + \beta \frac{\|Gy - TG(Gy), a\|\|Gy - TG(Gx), a\|\|Gx - TG(Gy)\| + \|Gx-Gy,a\|^3}{\|Gx-Gy,a\|^2}$$

$$+\gamma\left[\frac{\|Gx - TG(Gx), a\| + \|Gy - TG(Gy), a\|}{2}\right] + \delta\left[\frac{\|Gx - TG(Gx), a\| + \|G(y) - TG(Gx), a\|}{2}\right]$$

$$+\eta\|Gx-Gy,a\|$$

Taking $G(x) = p$, $G(y) = q$, *where* $p \neq q$

$$\geq \alpha \frac{\|p - TG(p), a\|\|q - TG(q), a\|}{\|p-q,a\|} + \beta \frac{\|q - TG(q), a\|\|q - TG(p), a\|\|p - TG(q)\| + \|p-q,a\|^3}{\|p-q,a\|^2}$$

$$+\gamma\left[\frac{\|p - TG(p), a\| + \|q - TG(q), a\|}{2}\right] + \delta\left[\frac{\|p - TG(q), a\| + \|q - TG(p), a\|}{2}\right] + \eta\|p-q,a\|$$

Taking $TG = R$ we get

$$\|R(p) - R(q), a\| \geq \alpha \frac{\|p - R(p), a\|\|q - R(q), a\|}{\|p-q,a\|} + \beta \frac{\|q - R(q), a\|\|q - R(p), a\|\|p - R(q)\| + \|p-q,a\|^3}{\|p-q,a\|^2}$$

$$+\gamma\left[\frac{\|p - R(p), a\| + \|q - R(p), a\|}{2}\right] + \delta\left[\frac{\|p - R(q), a\| + \|q - R(p), a\|}{2}\right] + \eta\|p-q,a\|$$

It is clear by theorem (1.1); that $R = TG$ has at least one fixed point say x_0 in K that is $R(x_0) = TG(x_0) = x_0$

And so $T.(TG)(x_0) = T(x_0)$

Or $T^2(Gx_0) = T(x_0)$

$G(x_0) = T(x_0)$

Now

$$\|Tx_0 - x_0, a\| = \|Tx_0 - T^2(x_0), a\| = \|Tx_0 - T.T(x_0), a\|$$

$$\geq \alpha \frac{\|G(x_0) - T(x_0), a\|\|GT(x_0) - T(Tx_0), a\|}{\|G(x_0) - G(Tx_0), a\|}$$

$$+ \beta \frac{\|G(Tx_0) - T(Tx_0), a\|\|G(Tx_0) - T(x_0), a\|\|G(x_0) - T(Tx_0), a\| + \|G(x_0) - G(Tx_0), a\|^3}{\|G(x_0) - G(Tx_0), a\|^2}$$

$$+ \gamma \left[\frac{\|G(x_0) - T(x_0), a\| + \|G(Tx_0) - T(Tx_0), a\|}{2}\right] + \delta \left[\frac{\|G(x_0) - T(Tx_0), a\| + \|G(Tx_0) - T(x_0), a\|}{2}\right]$$

$$+ \eta \|G(x_0) - G(Tx_0), a\|$$

$$= (\beta + \delta + \eta)\|Tx_0 - x_0, a\|$$

So $T(x_0) = x_0 \qquad (\beta + \gamma + \eta > 1)$

That is x_0 is the fixed point of T.

But $T(x_0) = G(x_0)$ so $G(x_0) = x_0$.

Hence x_0 is the fixed point of T and G.

Uniqueness:

If possible let $y_0 \neq x_0$ is another common fixed point of T and G.

Then $\|x_0 - y_0, a\| = \|T^2(x_0) - T^2(y_0), a\| = \|T(T(x_0)) - T(T(y_0)), a\|$

$$\geq \alpha \frac{\|G(Tx_0) - T(Tx_0), a\|\|G(Ty_0) - T(Ty_0), a\|}{\|G(Tx_0) - G(Ty_0), a\|}$$

$$+ \beta \frac{\|G(Ty_0) - T(Ty_0), a\|\|G(Ty_0) - T(Tx_0), a\|\|G(Tx_0) - T(Ty_0), a\| + \|G(Tx_0) - G(Ty_0), a\|^3}{\|G(Tx_0) - G(Ty_0), a\|^2}$$

$$+ \gamma \left[\frac{\|G(Tx_0) - T(Tx_0), a\| + \|G(Ty_0) - T(Ty_0), a\|}{2}\right] + \delta \left[\frac{\|G(Tx_0) - T(Ty_0), a\| + \|G(Ty_0) - T(Tx_0), a\|}{2}\right]$$

$$+ \eta \|G(Tx_0) - G(Ty_0), a\|$$

$$\geq \beta \|x_0 - y_0, a\| + \delta \|x_0 - y_0, a\| + \eta \|x_0 - y_0, a\|$$

$$\geq (\beta + \delta + \eta)\|x_0 - y_0, a\|$$

But $\beta + \delta + \eta > 1$

So $x_0 = y_0$. So common fixed point in unique.

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