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Solution of Poisson Equation with Mixed Boundary Conditions in Irregular Domain

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ABSTRACT: *This work presents a study of two finite element solutions using either linear triangular elements or bilinear rectangular elements. The mesh of triangular elements is obtained by dividing each quadrilateral element into two triangular elements or vice versa, the mesh of rectangular elements is obtained by merging two adjacent triangular elements. To assess performance, we use two dimensional steady state problems in an L-shaped plate as a test case.By comparing the two finite element solutions using linear triangular elements or bilinear rectangular elements, we observe that the rectangular elements produced more accurate solution for the temperature distribution for the L-shaped plate.*

KEYWORDS:*Triangular elements, Bilinear Quadrilateral elements, FEM*

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1. INTRODUCTION

Finite element method enjoys a firm theoretical foundation that is mostly free of ad hoc schemes and heuristic numerical approximations, thereby inspiring confidence in the physical relevance of the solution Pozrikidis (2014). A very popular approach for discretizing Partial Differential Equations (PDE), the finite element method, is based on variational forms (Adams and Fournier (2003); Zuger, (2013) and Evans (2010)).The crucial element in this method is that it uses a variational problem over a domain of the PDE.The practical implementation of the Finite element method(FEM) involves the determination of three ingredients that are closely related: nodes, elements, and basis functions. FEM nodes are points inside the domain and on its boundary that are used in the definition of the basis functions. The elements are relatively small subdivisions of the domain whose size and geometry is determined by the node locations (e.g., such that nodes coincide with element vertices).The FEM utilizes discrete elements to obtain the approximate solution of the governing differential equation. The final FEM system equation is constructed from the discrete element equations.

Aklilu, (2011) studied finite element and finite difference methods for elliptic and parabolic differential equations. He discussed the distinctiveness of the solution of an elliptic equation which was dependent on the boundary condition.He concluded that the number of elements has effect in quality of the FEM solution.

Pal and Kristian, (2004) worked on implementation of Finite Element Method for Poisson equation on a regular domain. Finite Elements Method for the Poisson equation was implemented using MATLAB,the use of GUI makes the program more handy because the linear system resulting from the FEM problem is sparse, symmetric and positive definite, the use of the preconditioned conjugate gradient method reduces the total required computational time.It was concluded that the FEM solution is obviously converging to the true solution.

Peiro and Sherwin, (2005) who studied finite difference, finite element and finite volume methods for partial differential equations asserted that the discretization of linear elliptic equations with either Finite Difference, Finite Element or Finite Volume methods leads to non-singular systems of equations that can easily solved by standard methods of solution.

Agbezuge, (2012) studied finite element solution of the Poisson equation with Dirichlet boundary conditions in a rectangular domain. The concepts utilized in solving the problem are weak formulation of the Poisson Equation, creation of a Finite Element Model on the basis of an assumed approximate solution, creation of 4-node rectangular elements by using interpolation functions of the Lagrange type, assembly of element equations, solution and post-processing of the results.

Patil and Prasad, (2013) studied numerical solution for two dimensional Laplace equation with Dirichlet boundary conditions in a rectangular domain. Numerical techniquesadopted are finite difference method (FDM),finite element method and Markov chain method (MCM) using spreadsheets. The numerical solutions obtained by FDM, FEM and MCM are compared with exact solution to check the accuracy of the developed scheme. They concluded that the power of the FEM becomes more evident, because the Finite Difference method will have much more difficulty in solving problems in a domain with complex geometries.

The motivation behind this work stems from the perception that, approximately the same element size, a mesh of rectangular elements would consist of approximately half as many elements as a mesh of triangular elements (a rectangular element may be found from merging two adjacent triangular elements). Furthermore, the number of edges in the rectangular mesh is approximately two-thirds that of the triangular mesh. For weighted residual methods, evaluating the boundary integral is one of the major computational costs. From these observations, one might expect that the use of quadrilateral elements may improve the computational efficiency of the schemes. The purpose of this paper is to compare the two finite element solutions using either linear triangular elements or bilinear rectangular elements.

In this paper, weak formulation of a weighted residual method for Laplace's equation is first summarized. Subsequently, discretization of the domain is performed using selected two-dimensional finite elements. We also discuss the strategy we to efficiently evaluate the elemental matrices required in the scheme. Lastly, we present the results of numerical solutions on a temperature distribution for L-shaped plate in Section 3, and draw some conclusions of the study in Section 4.

2.METHODOLOGY

2.1 The Governing Equation

The use of high numerical methods for the computational solution of Laplacian problems is significant in many fields of physics and engineering (Durojaye et al., 2019). The governing differential equation for Laplace's equation is

$$\nabla^2 u = 0 \quad 2.1$$

while Poisson's equation is

$$\nabla^2 u = f \quad 2.2$$

We shall consider Poisson's equation in the following formulation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ in } \Omega$$

for the two-dimensional domain Ω. The boundary conditions are

$$u = u_D \text{ on the boundary } \Gamma_D \quad 2.3$$

and

$$\frac{\partial u}{\partial n} = g \text{ on the boundary } \Gamma_N \quad 2.4$$

whereu_D and g denote known variable and flux boundary conditions, and *n* in Eq. (2.4) is the outward normal unit vector at the boundary.

The integral of weighted residual of the partial differential equation and boundary condition is

$$I = \int_\Omega v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f(x, y)\right) d\Omega - \int_{\Gamma_D} v \frac{\partial u}{\partial n} d\Gamma \quad 2.5$$

In order to develop the weak formulation of (2.5), integration by part is applied to reduce the order of differentiating within the integral. Manipulating the first integral on the right hand side of Eq. (2.5), we obtain,

$$\int_\Omega v\left(\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y}\right) d\Omega = -\int_\Omega \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right) d\Omega + \oint_\Gamma v\left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y\right) d\Gamma \quad 2.6$$

in which $n_x$is the x-component and $n_y$is the y-component of the unit vector which is assumed to be positive in the outward direction. Since the boundary integral can be written as

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \quad 2.7$$

Now, we can rewrite Eq. (2.6) as

$$\int_{\Omega} v\left(\frac{\partial^2 u}{\partial x}+\frac{\partial^2 u}{\partial y}\right)d\Omega=-\int_{\Omega}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)d\Omega+\oint_{\Gamma} v\frac{\partial u}{\partial n}d\Gamma \quad 2.8$$

For simplicity, the symbol $\oint$ to denote the line integral around a closed boundary is replaced by $\int$ We use Eq. (2.8) to Eq. (2.5) results in

$$I=-\int_{\Omega}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)d\Omega-\int_{\Omega} vf(x,y)d\Omega+\int_{\Gamma_N} v\frac{\partial u}{\partial n}d\Gamma \quad 2.9$$

Eq. (2.9) represents the weak formulation of equation (2.2). The first volume integral of Eq. (2.9) becomes a matrix term while the second volume integral and the line integral become a vector term.

2.1Discretisation of the Domain

For the purpose of this work, the problem domain is discretized into finite elements using linear triangular elements and bilinear rectangular elements.

2.2 Linear Triangular Element

The discretization of the domain in Eq. (2.2) is performed using selected 2-D finite elements. Linear triangular element has three nodes at the vertices of the triangle and the variable interpolation within the element is linear in x and y which is

$$u=\phi_1+\phi_2 x+\phi_3 y \quad 2.10$$

where ϕ_i is the constant to the determined. The interpolation function, Eq. (2.10), represents the nodal variables at the three nodal points. The values of x and y are derived by substituting it at each nodal point gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}=\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \quad 2.11$$

where x_i and y_i are the coordinate values at the ith node and $u_i$is the nodal variable.

Inverting the matrix and rewriting Eq. (2.11) give

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}=\frac{1}{2A}\begin{bmatrix} x_2y_3-x_3y_2 & x_3y_1-x_1y_3 & x_1y_2-x_2y_1 \\ y_2-y_3 & y_3-y_1 & y_1-y_2 \\ x_3-x_2 & x_1-x_3 & x_2-x_1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad 2.12$$

where

$$A=\frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \quad 2.13$$

The value of element node numbering in the counter-clockwise direction will be positive and negative otherwise. The element nodal sequence must be in the same direction for every element in the domain. Substitution of Eq. (2.12) into Eq. (2.10) we have

$$u=\psi_1(x,y)u_1+\psi_2(x,y)u_2+\psi_3(x,y)u_3 \quad 2.14$$

where $\psi_i(x,y)$ is the shape function for the linear triangular element and it is given below as:

$$\psi_1=\frac{1}{2A}[(x_2y_3-x_3y_2)+(y_2-y_3)x+(x_3-x_2)y] \quad 2.15$$

$$\psi_2=\frac{1}{2A}[(x_3y_1-x_1y_3)+(y_3-y_1)x+(x_1-x_3)y] \quad 2.16$$

$$\psi_3=\frac{1}{2A}[(x_1y_2-x_2y_1)+(y_1-y_2)x+(x_2-x_1)y] \quad 2.17$$

These shapes functions satisfy the conditions

$$\psi_i\left(x_j, y_j\right) = \delta_{ij} \quad 2.18$$

$$\text{and} \sum_{i=1}^{3} \psi_i = 1 \quad 2.19$$

Here, δ_{ij} is the Kronecker delta. That is,

$$\delta_{ij} = \begin{Bmatrix} 1 & if & i = j \\ 0 & if & i \neq j \end{Bmatrix} \quad 2.20$$

2.3 Element Matrix

The element matrix is computed as derived below:

$$\left[K^e\right] = \int_{\Omega^e} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) d\Omega$$

$$= \int_{\Omega^e} \left(\begin{Bmatrix} \frac{\partial \psi_1}{\partial x} \\ \frac{\partial \psi_2}{\partial x} \\ \frac{\partial \psi_3}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_3}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial y} \\ \frac{\partial \psi_3}{\partial y} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_3}{\partial y} \end{Bmatrix} \right) d\Omega^{2.21}$$

where Ω^e is the element domain.

Carrying out the integration after substituting the shape functions Eq (2.15) through (2.18) into Eq. (2.21) we have

$$\left[K^e\right] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} 2.22$$

where $k_{11} = \frac{1}{4A}[(x_3 - x_2)^2 + (y_2 - y_3)^2] \quad 2.23$

$$k_{12} = \frac{1}{4A}[(x_3 - x_2)(x_2 - x_3) + (y_2 - y_3)(y_3 - y_1)] \; 2.24$$

$$k_{13} = \frac{1}{4A}[x_3 - x_2)(x_2 - x_1) + (y_2 - y_3)(y_1 - y_2)] \; 2.25$$

$$k_{22} = \frac{1}{4A}[(x_1 - x_3)^2 + (y_3 - y_1)^2] \; 2.26$$

$$k_{33} = \frac{1}{4A}[(x_2 - x_1)^2 + (y_1 - y_2)^2] \; 2.27$$

$$k_{11} = k_{12}, k_{31} = k_{13}, k_{32} = k_{23} \; 2.28$$

The integral term of the other domain to be evaluated in Eq. (2.2) is

$$\int_{\Omega} vf(x, y) d\Omega \; 2.29$$

The computation of this integral over each linear triangular element results in a column vector which is

$$\int_{\Omega^e} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} f(x,y)d\Omega \quad 2.30$$

2.4 Bilinear Quadrilateral Element

The shape functions for the rectangular element can be derived from the interpolation function:

$$u = \phi_1 + \phi_2 x + \phi_3 y + \phi_4 xy \quad 2.31$$

This function is linear in both x and y. The shape functions can be derived by applying the same procedure as used above results in

$$\psi_1 = \frac{1}{4ab}(a-x)(b-y) \quad 2.32$$

$$\psi_2 = \frac{1}{4ab}(a+x)(b-y) \quad 2.33$$

$$\psi_3 = \frac{1}{4ab}(a+x)(b+y) \quad 2.34$$

$$\psi_4 = \frac{1}{4ab}(a-x)(b+y) \quad 2.35$$

where 2a and 2b are the length and height of the element, respectively.

2.5 Element Matrix Using Bilinear Shape functions.

The element matrix is computed as derived below:

$$\left[K^e\right] = \int_{\Omega^e} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) d\Omega$$

$$= \int_{\Omega^e} \left(\begin{Bmatrix} \frac{\partial \psi_1}{\partial x} \\ \frac{\partial \psi_2}{\partial x} \\ \frac{\partial \psi_3}{\partial x} \\ \frac{\partial \psi_4}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_3}{\partial x} & \frac{\partial \psi_4}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial y} \\ \frac{\partial \psi_3}{\partial y} \\ \frac{\partial \psi_4}{\partial y} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_3}{\partial y} & \frac{\partial \psi_4}{\partial y} \end{Bmatrix} \right) d\Omega \quad 2.36$$

where ψ_i is the bilinear shape functions.

Carrying out the integration for all terms, we have the following element matrix for bilinear rectangular element

$$\left[K^e\right] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} 2.37$$

where

$$k_{11} = \int_{-a}^{a}\int_{-b}^{b} \left[\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{\partial \psi_1}{\partial y} \right]$$

$$=\int_{-a}^{a}\int_{-b}^{b}\frac{1}{16a^2b^2}\left[(y-b)^2+(x-a)^2\right]dydx$$ 2.38
$$=\frac{a^2+b^2}{3ab}$$

$$k_{12}=\frac{a^2-2b^2}{6bc}$$ 2.39

$$k_{13}=-\frac{a^2+b^2}{6bc}$$ 2.40

$$k_{14}=\frac{b^2-2a^2}{6bc}$$ 2.41

$$k_{22}=k_{11},k_{23}=k_{14},\ k_{24}=k_{13},\ k_{33}=k_{11},\ k_{34}=k_{12},\ k_{44}=k_{11}$$ 2.42

The computation of this integral over each linear rectangular element results in a column vector which is

$$\int_{\Omega^e}\begin{Bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{Bmatrix}f(x,y)d\Omega$$ 2.43

The boundary integral in Eq. (1.9) is

$$\int_{\Gamma_N} v\frac{\partial u}{\partial n}d\Gamma=\sum\int_{\Gamma^e} v\frac{\partial u}{\partial n}d\Gamma$$ 2.44

Here Γ_N denotes the natural boundary and Γ^e shows the element boundary. The $\sum$ is taken over the elements which are located at the boundary of the domain and whose elements are subjected to natural boundary condition. Linear one- dimensional shape functions are used to interpolate the element boundary. The boundary integral along the element boundary becomes

$$\int_{\Gamma^e} v\frac{\partial u}{\partial n}d\Gamma=\bar{q}\int_{x_i}^{x_j}\begin{Bmatrix}\frac{x_j-x}{x_j-x_i}\\ \frac{x-x_i}{x_j-x_i}\end{Bmatrix}dx=\frac{\bar{q}h_{ij}}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$ 2.45

where $h_{ij}=x_j-x_i$: length of the element boundary.

3. NUMERICAL SOLUTIONS

In this section we solved some tests problems for two dimensional steady state problems. Both linear triangular and bilinear rectangular elements are used.

Test problem 1

Compute the temperature distribution for the L-shaped plate in Fig. 3.1. The boundary condition is also in the figure.

Solution

u(x, y) is the steady state temperature distribution in the domain. Fig. 3.1 has the boundary values given in that figure.

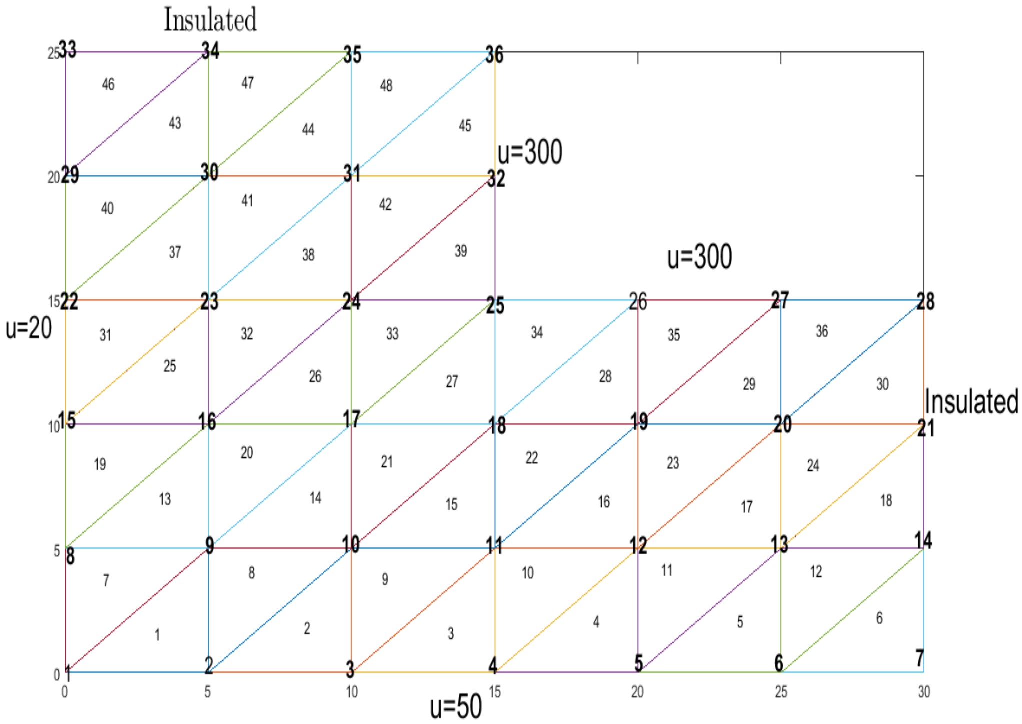

Figure 3.1

In order to solve this equation the Finite Element Method triangular elements were used, Table 1 shows the coordinates of nodes and the node numbers of the triangular subregions. We located 22 boundary points and 14 interior points, number them and divided the domain into 48 triangular subregions as depicted in Fig. 3.1. Table 2 showing coordinates of nodes and the node numbers of the subregions.

Table 2 Coordinates of Nodes and the Node Numbers of the Subregions from Fig. 3.1

Coordinates of Nodes				Node Numbers of Subregions			
Node	x	y		Elemen	Node1	Node2	Node3
1	0	0		1	1	2	9
2	5	0		2	2	3	10
3	10	0		3	3	4	11
4	15	0		4	4	5	12
5	20	0		5	5	6	13
6	25	0		6	6	7	14
7	30	0		7	1	9	8
8	0	5		8	2	10	9
9	5	5		9	3	11	10
10	10	5		10	4	12	11
11	15	5		11	5	13	12
12	20	5		12	6	14	13
13	25	5		13	8	9	16
14	30	5		14	9	10	17
15	0	10		15	10	11	18
16	5	10		16	11	12	19
17	10	10		17	12	13	20
18	15	10		18	13	14	21
19	20	10		19	8	16	15
20	25	10		20	9	17	16
21	30	10		21	10	18	17
22	0	15		22	11	19	18
23	5	15		23	12	20	19
24	10	15		24	13	21	20
25	15	15		25	15	16	23
26	20	15		26	16	17	24
27	25	15		27	17	18	25
28	30	15		28	18	19	26
29	0	20		29	19	20	27
30	5	20		30	20	21	28
31	10	20		31	15	23	22
32	15	20		32	16	24	23
33	0	25		33	17	25	24
34	5	25		34	18	26	25
35	10	25		35	19	27	26
36	15	25		36	20	28	27
				37	22	23	30
				38	23	24	31
				39	24	25	32
				40	22	30	29
				41	23	31	30
				42	24	32	31
				43	29	30	34
				44	30	31	35
				45	31	32	36
				46	29	34	33
				47	30	35	34
				48	31	36	35

Assembly of Global Matrix

The global coefficient matrix is determined by the coordinates of the vertices of the corresponding element. Since there are 36 nodes, the global coefficient matrix will be a 36 x 36 matrix. The assembly procedure was executed by writing a short Matlab® code, whose output is shown below:

For typical triangular element:

$$\left[k^e\right] = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

The global coefficient matrix for temperature distribution on the L-shaped plate in Fig. 4.1 with an insulated upper and right hand side edge was assembled as follows:

[K]=

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-0.5	0	0	0	0	0	-1	2	0	0	0	0	0	0	-0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	0	0	0	0	0	-1	2	0	0	0	0	0	0	-0.5	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	-1	4	-1	0	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	-0.5	2	-0.5	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	-0.5	2	-0.5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The assembled source and flux terms

{K}=[20,50,50,50,50,50,0,20,0,0,0,0,0,0,20,0,0,0,0,0,0,20,0,0,300,300,300,0,20,0,0,300,20,0,0,300]'

The assembled equation is of the form

[K]{U} = {F}

Having the Essential Boundary conditions and Natural boundary conditions on the boundary of the domain, the nodal solution vector should be of the form

{U}=[20,50,50,50,50,50,U7,20,U9,U10,U11,U12,U13,U14,20,U16,U17,U18,U19,U20,U21,20,U23,U24,300,300,300,U28,20,U30,U31,300,20,U34,U35,300]

so that the unknown values of {U} occur at global nodes 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 23, 24, 28, 30, 31, 34 and 35.

Note that because of the derivative boundary condition, we have to account for the six unknown temperatures along the plate's upper and right hand side edge. The finite element solution of this equation yields

Node	x	y	Solution
7	30	0	95.07285
9	5	5	59.09349
10	10	5	89.06929
11	15	5	113.0949
12	20	5	125.0569
13	25	5	131.303
14	30	5	140.1457
16	5	10	77.30467
17	10	10	134.0888
18	15	10	188.2534
19	20	10	205.8299
20	25	10	210.0092
21	30	10	202.904
23	5	15	96.03645
24	10	15	181.7277
28	30	15	251.452
30	5	20	105.1135
31	10	20	196.7855
34	5	25	107.6319
35	10	25	200.3007

Test problem 2

We solve the same test problem using bilinear rectangular elements.

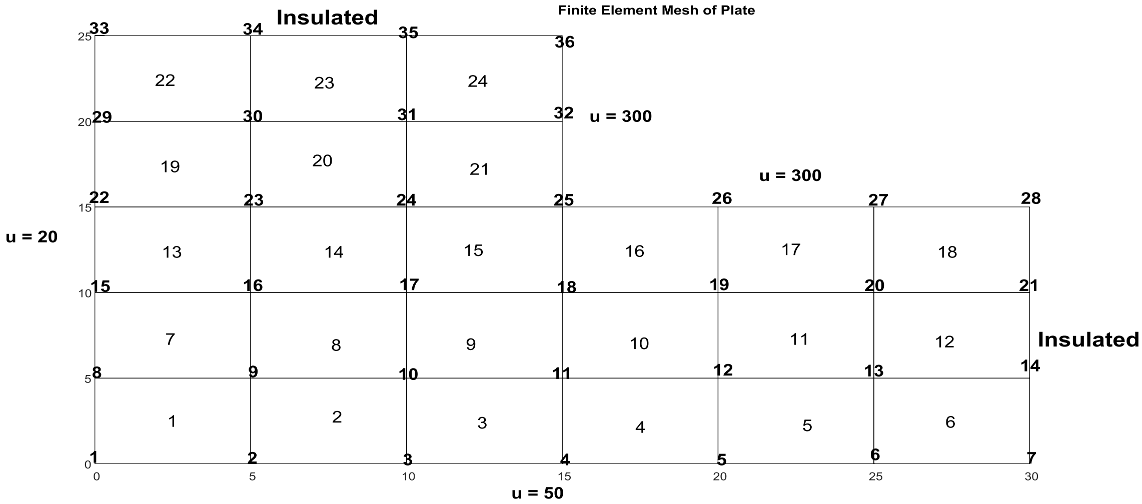

Figure 3.2

In order to solve this equation the Finite Element Method rectangular elements were used, Table 3.2 shows the node numbers of the rectangular subregions. We located 22 boundary points and 14 interior points, number them and divided the domain into 24 rectangular subregions as depicted in Fig. 3.2.
Table 3.2 showing node numbers of the Sub-regions.

Node Numbers of Subregions				
Element	**Node 1**	**Node 2**	**Node 3**	**Node 4**
1	1	2	9	8
2	2	3	10	9
3	3	4	11	10
4	4	5	12	11
5	5	6	13	12
6	6	7	14	13
7	8	9	16	15
8	9	10	17	16
9	10	11	18	17
10	11	12	19	18
11	12	13	20	19
12	13	14	21	20
13	15	16	23	22
14	16	17	24	23
15	17	18	25	24
16	18	19	26	25
17	19	20	27	26
18	20	21	28	27
19	22	23	30	29
20	23	24	31	30
21	24	25	32	31
22	29	30	34	33
23	30	31	35	34
24	31	32	36	35

Assembly of Global Matrix

The global coefficient matrix is determined by the coordinates of the vertices of the corresponding element. Since there are 36 nodes, the global coefficient matrix will be a 36 x 36 matrix. The assembly procedure was executed by writing a short Matlab® code, whose output is shown below.

For typical rectangular element:

$$\left[k^e\right]=\begin{bmatrix} 0.6667 & -0.1667 & -0.3333 & -0.1667 \\ -0.1667 & 0.6667 & -0.1667 & -0.3333 \\ -0.3333 & -0.1667 & 0.6667 & -0.1667 \\ -0.1667 & -0.3333 & -0.1667 & 0.6667 \end{bmatrix}$$

The global coefficient matrix was assembled as follows:

[K] =

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-0.333	-0.167	0	0	0	0	0	-0.333	1.3333	0	0	0	0	0	-0.333	-0.167	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.167	0	0	0	0	0	-0.333	1.3333	0	0	0	0	0	-0.333	-0.167	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.167	-0.333	0	0	0	0	0	1.3333	-0.333	0	0	0	0	0	-0.167	-0.333	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	-0.333	-0.333	-0.333	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	0	0	0	-0.333	2.6667	-0.333	0	-0.333	-0.333	-0.333
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	-0.167	1.3333	-0.167	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.333	-0.333	-0.333	0	-0.167	1.3333	-0.167
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The assembled source and flux terms

{K}=[20,50,50,50,50,50,0,20,0,0,0,0,0,0,20,0,0,0,0,0,0,20,0,0,300,300,300,0,20,0,0,300,20,0,0,300]'

The assembled equation is of the form

[K]{U} = {F}

Having the Essential Boundary conditions and Natural boundary conditions on the boundary of the domain, the nodal solution vector should be of the form

{U}=[20,50,50,50,50,50,U7,20,U9,U10,U11,U12,U13,U14,20,U16,U17,U18,U19,U20,U21,20,U23,U24,300,300,300,U28,20,U30,U31,300,20,U34,U35,300]

so that the unknown values of {U} occur at global nodes 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 23, 24, 28, 30, 31, 34 and 35.

The finite element solution of this equation yields

Node	x	y	Solution
7	30	0	115.6086
9	5	5	57.40073
10	10	5	88.53091
11	15	5	110.4661
12	20	5	123.6836
13	25	5	137.0492
14	30	5	138.3358
16	5	10	75.88347
17	10	10	134.7914
18	15	10	179.7056
19	20	10	207.0169
20	25	10	205.231
21	30	10	206.5176
23	5	15	93.1887
24	10	15	173.156
28	30	15	229.2449
30	5	20	104.1727
31	10	20	197.506
34	5	25	108.0987
35	10	25	201.432

By comparing the two finite element solutions using either linear triangular element or bilinear rectangular elements as shown below, we see that the rectangular elements produced more accurate solution in the present test problem.

Finite Element Solutions Using Linear and Bilinear Elements				
Node	x	y	Triangular element	Rectangular elements
1	0	0	20.0000	20.0000
2	5	0	50.0000	50.0000
3	10	0	50.0000	50.0000
4	15	0	50.0000	50.0000
5	20	0	50.0000	50.0000
6	25	0	50.0000	50.0000
7	30	0	95.0729	115.6086
8	0	5	20.0000	20.0000
9	5	5	59.0935	57.4007
10	10	5	89.0693	88.5309
11	15	5	113.0949	110.4661
12	20	5	125.0569	123.6836
13	25	5	131.3030	137.0492
14	30	5	140.1457	138.3358
15	0	10	20.0000	20.0000
16	5	10	77.3047	75.8835
17	10	10	134.0888	134.7914
18	15	10	188.2534	179.7056
19	20	10	205.8299	207.0169
20	25	10	210.0092	205.2310
21	30	10	202.9040	206.5176
22	0	15	20.0000	20.0000
23	5	15	96.0364	93.1887
24	10	15	181.7277	173.1560
25	15	15	300.0000	300.0000
26	20	15	300.0000	300.0000
27	25	15	300.0000	300.0000
28	30	15	251.4520	229.2449
29	0	20	20.0000	20.0000
30	5	20	105.1135	104.1727
31	10	20	196.7855	197.5060
32	15	20	300.0000	300.0000
33	0	25	20.0000	20.0000
34	5	25	107.6319	108.0987
35	10	25	200.3007	201.4320
36	15	25	300.0000	300.0000

4. CONCLUSION

A primary basic approach is to observe if the numerical solutions of Finite Element Method using the linear triangular elements is better than bilinear rectangular elements on a temperature distribution for the L-shaped plate. Simple elements such as linear triangles and bilinear rectangular were employed for the finite-element mesh in two dimensions. By comparing the two finite element solutions using linear triangular elements or bilinear rectangular elements, we observe that the rectangular elements produced more accurate solution for the temperature distribution for the L-shaped plate. Some slight differences can be observed in both methods, especially when there is a great point charge variation. This finding corroborates with Pozrikidis (2014) that the discretizing process of mesh can influence the results, and rectangular element produced accurate results when compared with triangular elements. Similarly to the findings Ezeh and Enem (2012) on comparative study on use triangular and rectangular finite elements in analysis of deep beam, she concluded that results of the analysis show that the use of rectangular elements yielded results that are closer to the exact solutions than the triangular element model.

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F.O. Ogunfiditmi, et. al. "Solution of Poisson Equation with Mixed Boundary Conditions in Irregular Domain."*American Journal of Engineering Research (AJER),* vol. 9(06), 2020, pp. 40-52.