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Fifth Degree Hermittian Polynomial Shape Functions for the Finite Element Analysis of Clamped Simply Supported Euler – Bernoulli Beam

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ABSTRACT: *Fifth degree Hermittian polynomial shape functions were used in this work for the flexural analysis of Euler – Bernoulli beams with a prismatic cross-section. The analysis used the finite element stiffness method to generate the stiffness and load matrices for the problem. The beam of length l, considered was clamped at x = 0, and simply supported at x = l, and carried a linearly distributed transverse load on the longitudinal axis. The results showed that the fifth degree Hermittian polynomial shape function yielded exact solutions for the deflection, the bending moment and shear force distributions. The effectiveness of the use of fifth degree Hermittian polynomial shape functions in the finite element stiffness method to solve the flexural problem of a propped cantilever beam under linearly distributed transverse load was thus established.*
KEYWORDS: *Fifth degree Hermittian polynomial shape functions, Euler – Bernoulli beam, finite element method, stiffness equation, stiffness matrix.*

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I. INTRODUCTION

Beams are structural elements designed to support transverse loads by the development of flexure. Beam is a three dimensional structure with a longitudinal axis that is usually more than the transverse dimensions. Beams are supported in various ways; and the support conditions determine the statical determinacy or otherwise of the beam flexure problem [1, 2].

Statically determinate beams are those beams that can analysed for support reactions and internal forces using the equations of static equilibrium alone. Conversely, statically indeterminate beams cannot be solved for reactions and the internal force distributions using the equations of static equilibrium only. Additional considerations are usually used to solve statically indeterminate beam problems. Beams are widely used in buildings, bridges, machines, ships and aerospace, and aircraft structures as well as naval structures [3, 4].

Theories used to describe the flexural behaviour of beams are: Euler – Bernoulli beam theory [5], Timoshenko beam theory [6], Mindlin beam theory, Vlasov beam theory and refined/shear deformable beam theories [7, 8]. Euler – Bernoulli beam theory has been found to be satisfactory for thin (slender) beams but unsatisfactory in the flexural analysis of moderately thick and thick beams; since shear deformation plays a significant role in their flexural analysis/behaviour [9, 10]. Moderately thick and thick beams are analysed more accurately using Timoshenko, Vlasov, Mindlin and the refined shear deformable beam theories. The three dimensional theory of elasticity approach can also be used to analyse thick beams under transverse loads.

The fundamental assumptions of the Timoshenko, Vlasov, Mindlin and refined/shear deformable beam theories which make them suitable for the flexural analysis of moderately thick beams is that shear deformation is taken into account in the formulation of their governing equations, irrespective of whether variational calculus methods were adopted or equilibrium methods were adopted.

The focus of this work is the Euler – Bernoulli beam theory. Methods found in the literature for the flexural analysis of Euler – Bernoulli beams in order to solve the governing equations, and find support reactions and internal bending moments and shear force distributions are force methods, flexibility methods, energy methods [11, 12]. Specific methods include: slope – deflection method, method of consistent deformations, Macaulay's method of integration, method of singularity functions, moment distributions method, method of virtual displacements, method of virtual work etc.

In this work, fifth degree Hermittian polynomial functions are used for the flexural analysis of a statically indeterminate Euler – Bernoulli beam.

II. THEORETICAL FRAMEWORK

The Euler – Bernoulli beam theory assumes as follows:[7, 13]

(i) The beam has a longitudinal plane of symmetry, with the cross-section symmetric about this plane. Also, the loads and supports are symmetric about this longitudinal plane of symmetry. With these conditions, the beam will only undergo bending deformation and no twisting.

(ii) Cross-section which are plane and perpendicular to the axis of the undeformed beam remain plane and perpendicular to the deflection curve of the deformed beam. This is valid provided the beam is sufficiently long and slender.

(iii) Deformation in the transverse direction and hence transverse strain is considered insignificant and can be disregarded.

(iv) Beam material is linear elastic, isotropic and homogeneous.

(v) The beam is in general a three dimensional body with a fairly complex three dimensional stress state.

Since there are no forces in the z – direction, the beam can be considered to be in plane stress state and the stress – strain relations become:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \mu\sigma_{yy}) \quad (1)$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \mu\sigma_{xx}) \quad (2)$$

$$\varepsilon_{zz} = \frac{\mu}{E}(\sigma_{xx} + \sigma_{yy}) \quad (3)$$

$$\varepsilon_{xy} = \frac{1+\mu}{E}\sigma_{xy} \quad (4)$$

$$\varepsilon_{xz} = 0 \quad (5)$$

$$\varepsilon_{yz} = 0 \quad (6)$$

where σ_{xx}, σ_{yy} are normal stresses; σ_{xy} is the shear stress, ε_{xx}, ε_{yy}, ε_{zz} are normal strains; ε_{xy}, ε_{xz}, ε_{yz} are the shear strains; E is the Young's modulus of elasticity, and μ is the Poisson's ratio.

The assumption that the transverse normal stresses σ_{yy} are considered insignificant compared to the flexural stresses σ_{xx} reduces the stress strain relations to the one dimensional stress – strain equation

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \quad (7)$$

Stresses are related to the distances, y from the neutral axis by:

$$\sigma = -\frac{E}{R}y \quad (8)$$

where R is the radius of curvature, and E is the Young's modulus of elasticity.

The resultant force of the normal stress distribution over the cross-section must be zero, while the resultant moment of the normal stress distribution is M_{xx}, resulting in the equations:

$$\iint \sigma dA = -\frac{E}{R}\iint y dA = 0 = \frac{-E}{R}\bar{y}A \quad (9)$$

$$M = \iint \sigma y dA = -\frac{E}{R}\iint y^2 dA = \frac{-\sigma}{y}\iint y^2 dA \quad (10)$$

where $\bar{y}$ is the centroid of the cross-section.

$$\therefore \quad \bar{y} = 0 \quad (11)$$

The neutral axis passes through the centroid of the cross-section.

$$\sigma = \frac{-My}{I} \quad (12)$$

where $$I = \iint y^2 dA \tag{13}$$

I is the second moment of area or the moment of inertia of the cross-section about the neutral axis.

2.1 Research aim and objectives

The research aim is to use the fifth degree Hermittian polynomial functions as shape functions in a finite element analysis of statically indeterminate Euler – Bernoulli beam with prismatic cross-section. The specific objectives are:

(i) to formulate fifth degree Hermittian polynomials that satisfy general boundary conditions.
(ii) to express the deflection function in terms of the fifth degree Hermittian polynomial shape functions for a general Euler – Bernoulli beam problem.
(iii) to express the deflection function for the problem of Euler – Bernoulli beam of length l, with clamped support at $x = 0$ and simple support at $x = l$ in terms of fifth degree Hermittian polynomial shape function.
(iv) to find the element stiffness matrix and the load matrix for the statically indeterminate Euler – Bernoulli beam problem considered by evaluation of the appropriate integrations.
(v) to find the stiffness equation for the statically indeterminate Euler – Bernoulli beam considered.
(vi) to solve for the unknown deflection parameters of the deflection function by inversion of the stiffness equation and hence determine the deflection function.
(vii) to find the bending moment distribution along the longitudinal axis of the Euler – Bernoulli beam considered using the bending moment – deflection relations.
(viii) to find the shear force distribution along the longitudinal axis of the Euler – Bernoulli beam considered using the shear force – deflection relation.

III. METHODOLOGY

Fifth degree Hermittian polynomial $P_h(x)$ that satisfies the end conditions

$$P_{1h}(x=0)=1 \tag{14}$$
$$P_{1h}(x=l)=0 \tag{15}$$
$$P'_{1h}(x=0)=0 \tag{16}$$
$$P'_{1h}(x=l)=0 \tag{17}$$
$$P''_{1h}(x=0)=0 \tag{18}$$
$$P''_{1h}(x=l)=0 \tag{19}$$

can be found from the polynomial:

$$p(x) = a_0 + \frac{a_1 x}{l} + a_2\left(\frac{x}{l}\right)^2 + a_3\left(\frac{x}{l}\right)^3 + a_4\left(\frac{x}{l}\right)^4 + a_5\left(\frac{x}{l}\right)^5 \tag{20}$$

Thus,

$$P_{1h}(x) = \left(1 - 10\left(\frac{x}{l}\right)^3 + 15\left(\frac{x}{l}\right)^4 - 6\left(\frac{x}{l}\right)^5\right)w_0 \tag{21}$$

and $$w(x) = N_{1h}(x)w_0 \tag{22}$$

where $N_{1h}(x)$ is the fifth degree Hermittian polynomial shape function corresponding to the end conditions Equations (14 – 19).

Similarly for

$$P_{2h}(x=0)=0 \tag{23}$$
$$P_{2h}(x=l)=0 \tag{24}$$
$$P'_{2h}(x=0)=1 \tag{25}$$
$$P'_{2h}(x=l)=0 \tag{26}$$
$$P''_{2h}(x=0)=0 \tag{27}$$
$$P''_{2h}(x=l)=0 \tag{28}$$

We find

$$P_{2h}(x) = \left(\frac{x}{l} - 6\left(\frac{x}{l}\right)^3 + 8\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5\right)lw'_0 \tag{29}$$

For,

$$P_{3h}(x=0)=0 \tag{30}$$
$$P_{3h}(x=l)=0 \tag{31}$$
$$P'_{3h}(x=0)=0 \tag{32}$$
$$P'_{3h}(x=l)=0 \tag{33}$$
$$P''_{3h}(x=0)=1 \tag{34}$$
$$P''_{3h}(x=l)=0 \tag{35}$$

We have

$$P_{3h}(x)=\left(\frac{1}{2}\left(\frac{x}{l}\right)^2-\frac{3}{2}\left(\frac{x}{l}\right)^3+\frac{3}{2}\left(\frac{x}{l}\right)^4-\frac{1}{2}\left(\frac{x}{l}\right)^5\right)l^2w_0'' \quad (36)$$

For,

$P_{4h}(x=0)=0$ (37) $\quad P_{4h}(x=l)=1$ (38)

$P'_{4h}(x=0)=0$ (39) $\quad P'_{4h}(x=l)=0$ (40)

$P''_{4h}(x=0)=0$ (41) $\quad P''_{4h}(x=l)=0$ (42)

We have

$$P_{4h}(x)=\left(10\left(\frac{x}{l}\right)^3-15\left(\frac{x}{l}\right)^4+6\left(\frac{x}{l}\right)^5\right)w_l \quad (43)$$

For

$P_{5h}(x=0)=0$ (44) $\quad P_{5h}(x=l)=0$ (45)

$P'_{5h}(x=0)=0$ (46) $\quad P'_{5h}(x=l)=1$ (47)

$P''_{5h}(x=0)=0$ (48) $\quad P''_{5h}(x=l)=0$ (49)

We have

$$P_{5h}(x)=\left(-4\left(\frac{x}{l}\right)^3+7\left(\frac{x}{l}\right)^4-3\left(\frac{x}{l}\right)^5\right)lw_l' \quad (50)$$

For

$P_{6h}(x=0)=0$ (51) $\quad P_{6h}(x=l)=0$ (52)

$P'_{6h}(x=0)=0$ (53) $\quad P'_{6h}(x=l)=0$ (54)

$P''_{6h}(x=0)=0$ (55) $\quad P''_{6h}(x=l)=1$ (56)

We have

$$P_{6h}(x)=\left(\frac{1}{2}\left(\frac{x}{l}\right)^3-\left(\frac{x}{l}\right)^4+\frac{1}{2}\left(\frac{x}{l}\right)^5\right)l^2w_l'' \quad (57)$$

The deflection function can be expressed as:

$$w(x)=P_{1h}+P_{2h}+P_{3h}+P_{4h}+P_{5h}+P_{6h} \quad (58)$$

$$w(x)=w^T N_n^T \quad (59)$$

$$(w)=\begin{pmatrix} w_0 \\ w_0' \\ w_0'' \\ w_l \\ w_l' \\ w_l'' \end{pmatrix} \quad (60)$$

4.0 Results

Case 1:

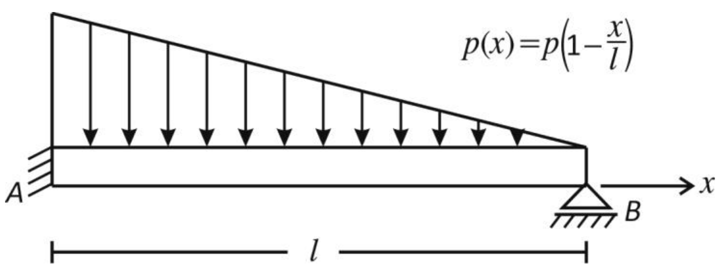

Figure 1: Statically indeterminate propped cantilever Euler – Bernoulli beam

For fixed simply supported Euler – Bernoulli beam, the boundary conditions are:

$$w(0) = w_0 = 0 \quad (61)$$

$$w'(0) = w'_0 = 0 \quad (62)$$

$$w_l = w(x = l) = 0 \quad (63)$$

$$M(l) = 0 \quad (64) \qquad w''_l = 0 \quad (65)$$

Then, the deflection function becomes

$$w(x) = N_{3h}(x)w''_0 + N_{5h}(x)w'_l \quad (66)$$

$$w(x) = N_{3h}(x)w''_0 + N_{5h}(x)\theta_l \quad (67)$$

$$w(x) = N_{3h}(x)\theta'_0 + N_{5h}(x)\theta_l \quad (68)$$

$$w(x) = \left(\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5\right)\theta'_0 + \left(-4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5\right)\theta_l \quad (69)$$

Here, θ'_0 and θ_l are the unknown generalised deflection parameters. The Hermittian polynomials are the shape functions

$$w(x) = N_1\theta'_0 + N_2\theta_l \quad (70)$$

where

$$N_1 = N_{3h} = \frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5 \quad (71)$$

$$N_2 = N_{5h} = -4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5 \quad (72)$$

The stiffness matrix is

$$k = \int_0^l EI\, N_u^T\, N_u^{iv}\, dx \quad (73)$$

where

$$N_u^{iv} = \frac{d^4}{dx^4}(N_u(x)) = (N_1^{iv}(x)\, N_2^{iv}(x)) \quad (74)$$

The load (force) vector is

$$F = \int_0^l p(x)N_u^T\, dx \quad (75)$$

Thus,

$$K = \int_0^l EI \begin{pmatrix} N_1(x) \\ N_2(x) \end{pmatrix} (N_1^{iv}(x) \;\; N_2^{iv}(x))\, dx \quad (76)$$

$$K = \int_0^l EI \begin{pmatrix} N_1(x)N_1^{iv}(x) & N_1(x)N_2^{iv}(x) \\ N_2(x)N_1^{iv}(x) & N_2(x)N_2^{iv}(x) \end{pmatrix} dx \quad (77)$$

The elements of the stiffness matrix are:

$$k_{11} = \int_0^l EI\, N_1(x)N_1^{iv}(x)\, dx \quad (78)$$

$$k_{12} = \int_0^l EI\, N_1(x)N_2^{iv}(x)\, dx \quad (79)$$

$$k_{21} = \int_0^l EI\, N_2(x)\; N_1^{iv}(x)\; dx \tag{80}$$

$$k_{22} = \int_0^l EI\, N_2(x)\; N_2^{iv}(x)\; dx \tag{81}$$

By differentiation,

$$N_1' = \frac{1}{2}\frac{2x}{l^2} - \frac{3}{2}\frac{3x^2}{l^3} + \frac{3}{2}\frac{4x^3}{l^4} - \frac{1}{2}\frac{5x^4}{l^5} \tag{82}$$

$$N_1'' = \frac{1}{l^2} - \frac{9}{2}\frac{2x}{l^3} + 6\cdot\frac{3x^2}{l^4} - \frac{5}{2}\cdot\frac{4x^3}{l^5} \tag{83}$$

$$N_1'' = \frac{1}{l^2} - \frac{9x}{l^3} + \frac{18x^2}{l^4} - \frac{10x^3}{l^5} \tag{84}$$

$$N_1''' = \frac{-9}{l^3} + \frac{36x}{l^4} - \frac{30x^2}{l^5} \tag{85}$$

$$N_1^{iv} = \frac{36}{l^4} - \frac{60x}{l^5} \tag{86}$$

$$N_2' = -4\cdot\frac{3x^2}{l^3} + \frac{28x^3}{l^4} - \frac{15x^4}{l^5} \tag{87}$$

$$N_2'' = \frac{24x}{l^3} + \frac{84x^2}{l^4} - \frac{60x^3}{l^5} \tag{88}$$

$$N_1''' = \frac{-24}{l^3} + \frac{168x}{l^4} - \frac{180x^2}{l^5} \tag{89}$$

$$N_1^{iv} = \frac{168}{l^4} - \frac{360x}{l^5} \tag{90}$$

$$\int_0^l N_1(x)\,N_1^{iv}(x)\,dx = \int_0^l \left(\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5\right)\left(\frac{36}{l} - \frac{60}{l^5}\right)dx \tag{91}$$

$$= \frac{1}{l^4}\int_0^l \left(\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5\right)\left(36 - \frac{60x}{l}\right)dx \tag{92}$$

$$= \frac{3}{35l^3} \tag{93}$$

$$\int_0^l N_1(x)\,N_2^{iv}(x)\,dx = \int_0^l \left(\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5\right)\left(\frac{168}{l^4} - \frac{360x}{l^5}\right)dx \tag{94}$$

$$= \frac{1}{l^4}\int_0^l \left(\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5\right)\left(168 - \frac{360x}{l}\right)dx \tag{95}$$

$$= \frac{4}{35l^3} \tag{96}$$

$$\int_0^l N_2(x)\,N_2^{iv}(x)\,dx = \int_0^l \left(-4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5\right)\left(\frac{168}{l^4} - \frac{360x}{l^5}\right)dx \tag{97}$$

$$= \frac{1}{l^4}\int_0^l \left(-4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5\right)\left(168 - \frac{360x}{l}\right)dx \tag{98}$$

$$= \frac{192}{35l^3} \tag{99}$$

$$\int_0^l N_2\, N_1^{iv}\, dx = \int_0^l \left(-4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5\right)\left(\frac{36}{l^4} - \frac{60x}{l^5}\right)dx \tag{100}$$

$$= \frac{1}{l^4}\int_0^l \left(-4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5\right)\left(36 - \frac{60x}{l}\right)dx \tag{101}$$

$$= \frac{4}{35l^3} \tag{102}$$

The force vector is

$$\vec{F} = \int_0^l p\left(1 - \frac{x}{l}\right) \begin{pmatrix} \frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right) + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5 \\ -4\left(\frac{x}{l}\right)^3 + 7\left(\frac{x}{l}\right)^4 - 3\left(\frac{x}{l}\right)^5 \end{pmatrix} dx \tag{103}$$

$$\vec{F} = \begin{pmatrix} \frac{1}{210}pl \\ \frac{-6}{210}pl \end{pmatrix} \tag{104}$$

$$\frac{EI}{l^3}\begin{pmatrix} \frac{3}{35} & \frac{4}{35} \\ \frac{4}{35} & \frac{192}{35} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{l}{210} \\ \frac{-6l}{210} \end{pmatrix} p \tag{105}$$

$$\frac{EI}{l^3}\begin{pmatrix} 3 & 4 \\ 4 & 192 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = pl\begin{pmatrix} \frac{1}{6} \\ \frac{-8}{6} \end{pmatrix} \tag{106}$$

$$\begin{pmatrix} 3 & 4 \\ 4 & 192 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{pl^4}{EI}\begin{pmatrix} \frac{1}{6} \\ \frac{-8}{6} \end{pmatrix} \tag{107}$$

Solving,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}\frac{pl^4}{120EI} = \begin{pmatrix} \frac{8}{120} \\ \frac{-1}{120} \end{pmatrix}\frac{pl^4}{EI} \tag{108}$$

Hence,

$$w(x) = \left(\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{3}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^4 - \frac{1}{2}\left(\frac{x}{l}\right)^5\right)\frac{8}{120}\frac{pl^4}{EI} -$$

$$\frac{pl^4}{120EI}\left(-4\left(\frac{x}{l}\right)^3+7\left(\frac{x}{l}\right)^4-3\left(\frac{x}{l}\right)^5\right) \quad (109)$$

$$w(x)=\frac{pl^4}{120EI}\left(4\left(\frac{x}{l}\right)^2-12\left(\frac{x}{l}\right)^3+12\left(\frac{x}{l}\right)^4-4\left(\frac{x}{l}\right)^5+4\left(\frac{x}{l}\right)^3-7\left(\frac{x}{l}\right)^4+3\left(\frac{x}{l}\right)^5\right) \quad (110)$$

$$w(x)=\frac{pl^4}{120EI}\left(4\left(\frac{x}{l}\right)^2-8\left(\frac{x}{l}\right)^3+5\left(\frac{x}{l}\right)^4-\left(\frac{x}{l}\right)^5\right) \quad (111)$$

This yields the exact solution for the deflection function.
The bending moment is found from the bending moment – deflection equation.

$$M(x)=-EI\,w''(x) \quad (112)$$

$$w'(x)=\frac{pl^4}{120EI}\left(4\frac{2x}{l^2}-24\frac{x^2}{l^3}+\frac{20x^3}{l^4}-\frac{5x^4}{l^5}\right) \quad (113)$$

$$w''(x)=\frac{pl^4}{120EI}\left(\frac{8}{l^2}-\frac{48x}{l^3}+\frac{60x^2}{l^4}-\frac{20x^3}{l^5}\right) \quad (114)$$

$$w''(x)=\frac{pl^2}{120EI}\left(8-48\left(\frac{x}{l}\right)+60\left(\frac{x}{l}\right)^2-20\left(\frac{x}{l}\right)^3\right) \quad (115)$$

$$w'''(x)=\frac{pl^2}{120EI}\left(\frac{-48}{l}+\frac{120x}{l^2}-\frac{60x^2}{l^3}\right) \quad (116)$$

$$w'''(x)=\frac{pl}{120EI}\left(-48+\frac{120x}{l}-60\left(\frac{x}{l}\right)^2\right) \quad (117)$$

$$M(x)=\frac{-pl^2}{120}\left(8-48\left(\frac{x}{l}\right)+60\left(\frac{x}{l}\right)^2-20\left(\frac{x}{l}\right)^3\right) \quad (118)$$

$$Q(x)=\frac{-pl}{120}\left(-48+120\left(\frac{x}{l}\right)-60\left(\frac{x}{l}\right)^2\right) \quad (119)$$

$$M(0)=\frac{-8pl^2}{120}=\frac{-pl^2}{15} \quad (120)$$

$$Q(0)=\frac{48pl}{120}=\frac{2pl}{5} \quad (121)$$

$$M(x=l)=0 \quad (122)$$

$$Q(x=l)=\frac{-pl}{120}(-48+120-60)=\frac{-pl}{10} \quad (123)$$

IV. DISCUSSION

Fifth degree Hermittian polynomials that satisfy the end conditions were used in this work to solve the statically indeterminate flexural problem of an Euler – Bernoulli beam under given transverse load distribution. The problem considered was an Euler – Bernoulli beam clamped at the left end $x = 0$, and simply supported at the right end $x = l$, where l is the length of the beam, and the x coordinate variable is used to define the longitudinal beam axis. The beam is subjected to a linear distribution of load applied transversely to the longitudinal axis.

The fifth degree Hermittian polynomial functions were used to express the unknown deflection function as Equation (58). The boundary conditions – Equations (61 – 65) of the specific problem considered were applied to the deflection function to obtain the simplified deflection function (also in terms of fifth degree Hermittian functions) that satisfied the boundary conditions as Equation (68) or (69). The displacement finite element method was used to express the stiffness matrix and the load matrix in terms of the fifth degree Hermittian polynomial shape functions as Equations (73) and (75) respectively. The stiffness matrix terms were evaluated using Equations (78) – (81). The load vector was evaluated using Equation (103) as Equation (104). The finite element stiffness equation was thus expressed as Equation (105). The stiffness equation was solved to obtain the unknown deflection parameters of the deflection function as Equation (108). The deflection was thus determined as Equation (111). It was observed that the deflection function obtained using the fifth degree Hermittian polynomial shape functions were the exact solution to the problem. The bending moment distribution was obtained from the bending moment – deflection relations as Equation (118). Similarly, the shear force distribution, obtained using the shear force deflection relation was given by Equation (119). It was observed that both the bending moment distribution and the shear force distribution have exact solutions for bending moment and shear force for the problem considered. The bending moment at the fixed support was found as Equation (120). The shear force at the fixed support was found as Equation (121). The shear force of the simple support was obtained as Equation (123).

V. CONCLUSION

The following conclusions are made from the study:

(i) Fifth degree Hermittian polynomial shape functions can be effectively used in the finite element displacement (stiffness) method to solve the flexural problem of statically – indeterminate Euler – Bernoulli beams.

(ii) Fifth degree Hermittian polynomial shape functions are constructed to aprori satisfy the relevant boundary conditions of the beam, and hence yield exact solutions for deflections, bending moments and shear force distributions within the theoretical framework of the Euler – Bernoulli beam theory.

(iii) The boundary value problem (BVP) of solving the ordinary differential equation of flexure of Euler – Bernoulli beams subject to the boundary conditions is converted to a problem of linear algebra where the unknown displacement parameters of the deflection function are the unknowns to be determined.

(iv) The use of numerical integration algorithms permits the extensions of the method to Euler – Bernoulli beams with non-prismatic cross-sections.

(v) Integration software tools can be used to evaluate the integration problems involved in the evaluation of the elements of the stiffness matrix and the load matrix.

REFERENCES

[1]. Ryder G.N. (1969). Strength of Materials. Third Edition Macmillan.
[2]. Sarkar B.K. (2007). Strength of Materials. Tata McGraw-Hill Publishing Company Limited, New Delhi.
[3]. Roca P., Cervera M. and Gariup G. (2010). Structural analysis of masonry historical constructions. Classical and Advanced Approaches, Archives of Computational Methods in Engineering Vol 17 No 3 pp. 299 – 325.
[4]. Module 7. Simple Beam Theory http//web.mit.edu.16.20/homepage/7_Simple Beam Theory/Simp.
[5]. The Bernoulli – Euler Beam https//www.colorado.edu/engineering/CAS/courses_ d/Avmmid/A.
[6]. Haque Aamer. Introduction to Timoshenko Beam Theory. http//www.clearly. impossible.com/a haque/timoshenko.pdf.
[7]. 7.4. The Elementary Beam Theory – Engineering. http/homepages.Engineering. Aukland.ac.nz/pke015/Solid Mechanics.
[8]. Bailey C., Bull T. and Laurence A. (2013). The bending of beams and the second moment of area. The Plymouth Student Scientist Vol 6 No 2 pp. 328 – 339.
[9]. Bansal R.K. (2005). A Textbook of Strength of Materials. Laxmi Publications (P) Ltd, New Delhi.
[10]. Lilkova – Markova S., Mladenski A. and Kindova – Petrora A. (2014). Selected topics on strength of materials. January 2014. http://nacy.bg/filebank/att_6146 pdf.
[11]. Ahmad S.W. Theory of Structures. Chapter 2. Deflection Macaulay Method. http//ocw.ump.edu.my/mod/resource/view.php?id – 277.
[12]. Livesley R.K. (1964). Matrix Methods of Structural Analysis. Ist Edition, Science Direct.
[13]. Chapter 5. http//personal.egr.uri.edu/sadd/mce565/ch.5.pdf.

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