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Applications Of Differential Transform Method To Integral Equations

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Abstract: *In this article, the Differential Transform Method is applied for solving integral equations. The approximate solution of the integral equation is calculated in the form of series with easily computable components. This powerful method catches the exact solution. Some integral equations are solved as examples. The results of the differential transform method is in good agreement with those obtained by using the already existing ones. The proposed method is promising to extensive class of linear and nonlinear problems.*
Keywords: *Differential Transform, Nonlinear phenomena, Integral equations.*

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I. INTRODUCTION

Nonlinear phenomena have important effects in applied mathematics, physics and related to engineering, many such physical phenomena are modeled in terms of nonlinear differential equations [3,4,6]. The concept of the differential transform was first proposed by Zhou [5]. Integral equations arise in many scientific and engineering problems[1,2].The potential theory contributed more than any field to give rise to integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping, and water waves also contributed to the creation of integral equations [1]. Many other applications in science and engineering are described by integral equations . The Volterra's population growth model [1], biological species living together, propagation of stocked fish in a new lake, the heat transfer and the heat radiation are among many areas that are described by integral equations [1]. In this paper we presented a variety of integral equations that were handled using differential transform method. The problems we handled led in most cases to the determination of exact solutions. Because it is not always possible to find exact solutions to problems of physical sciences, much work is devoted to obtaining qualitative approximations that high light the structure of the solution. It is the aim of this paper to handle some integral equations taken from a variety fields [7,8].

1.1 The Differential Transform Method

Suppose that the solution $u(x;t)$ is analytic at $(X;Y)$, then the solution $u(x;t)$ can be represented by the Taylor series[1].

$$u(x,t) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k_1! \dots k_n! \, h!} \left[\frac{\partial^{k_1+\dots+k_n+h} u(\tilde{x},\tilde{t})}{\partial x^{k_1} \dots \partial x^{k_n} \partial x^{h} u(\tilde{x},\tilde{t})} \right]$$

$$\left(\prod_{i=1}^{n} (x_i - \tilde{x}_i)^{k_i} \right) (t - \tilde{t})^{\,h} \qquad (1)$$

Definition 1.1

Let us define the $(n+1)$ dimensional differential transform $U(\tilde{k},\tilde{h})$ by

$$U(x,t) = \frac{1}{k_1!\dots k_n!\,h!}\left[\frac{\partial^{k_1+\dots+k_n+h}u(\tilde{x},\tilde{t})}{\partial x^{k_1}\dots\partial x^{k_n}\partial x^{h}u(\tilde{x},\tilde{t})}\right] \quad (2)$$

Definition 1.2

The differential inverse transform of $U(x,t)$ is define by $u(x,t)$ of the form in (1). Thus $u(x,t)$ can be written by:

$$u(x,t) = \sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)\left(\prod_{i=1}^{n}(x_i-\tilde{x}_i)^{k_i}\right)(t-\tilde{t})^{h} \quad (3)$$

An arbitrary function $f(x)$ can be expanded in Taylor series about a point $x = 0$ as:

$$f(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left\{\frac{d^k f}{dx^k}\right\}_{x=0} \quad (4)$$

The differential inverse transform of $u(x,t)$ is define by:

$$F(x) = \frac{1}{k!}\left\{\frac{d^k f(x)}{dx^k}\right\}_{x=0} \quad (5)$$

Then the inverse differential transform is:

$$F(x) = \sum_{k=0}^{\infty}x^k\ F(k) \quad (6)$$

3 The fundamental operation of Differential Transformation Method [1,3]:

(3.1) If $y(x) = g(x) \pm h(x)$ then $Y(k) = G(k) \pm H(k)$

$$F(k) = \frac{1}{k!}\left\{\frac{d^k f}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k y(x)}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k\big(g(x)\pm h(x)\big)}{dx^k}\right\}_{x=0}$$
$$= \frac{1}{k!}\left\{\frac{d^k g(x)}{dx^k} \pm \frac{d^k h(x)}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k g(x)}{dx^k}\right\}_{x=0} \pm \frac{1}{k!}\left\{\frac{d^k h(x)}{dx^k}\right\}_{x=0}$$
$$= G(x) \pm H(x)$$

(3.2) If $y(x) = \alpha g(x)$ then $Y(k) = \alpha G(x)$.
(3.3)If $y(x) = \frac{dg(x)}{dx}$ then Y(k)=(k+1)G(k),

$$Y(k) = \frac{1}{k!}\left\{\frac{d^k y(x)}{dx^k}\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^k}{dx^k}\left(\frac{d^k g(x)}{dx^k}\right)\right\}_{x=0} = \frac{1}{k!}\left\{\frac{d^{k+1}g(x)}{dx^{k+1}}\right\}_{x=0}$$
$$= \frac{1}{k!}(k+1)!\,G(k+1) = (k+1)G(k)$$

(3.4) If y(x)=$\frac{d^2 g(x)}{dx^2}$ then $Y(k) = (k+1)(k+2)G(k+1)$

(3.5) If y(x)=$\frac{d^m g(x)}{dx^m}$ then $Y(k) = (k+1)(k+2)\dots(k+m)G(k+m)$
(3.6) If $y(x) = 1$ then $Y(k) = \delta(k)$
(3.7) If $y(x) = x$ then $Y(k) = \delta(k-1)$
(3.8) If $y(x) = x^m$ then $Y(k) = \delta(k-m) = \begin{cases}1\ if\ k = m\\ 0\ if\ k \neq m\end{cases}$
(3.9) If $y(x) = g(x)h(x)$ then $Y(k) = \sum_{m=0}^{k}H(k)G(k-m)$
(3.10) If $y(x) = e^{\alpha x}$ then $Y(k) = \frac{\alpha^k}{k!}$
(3.11) If $y(x) = (1+x)^m$ then $Y(k) = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$

(3.12) If $y(x) = \sin(wx + \alpha)$, then $Y(k) = \frac{w^k}{k!}\sin(k\pi + \alpha)$ where w and α are constants.

(3.13) If $y(x) = \cos(wx + \alpha)$, then $Y(k) = \frac{w^k}{k!}\cos(k\pi + \alpha)$ where w and α are constants.

4 Applications

In this section , we apply the (DTM) to some to integral equations.
For the following problems, we need the following theorem.

4.1 Theorem

Suppose that $U(k)$ and $G(k)$ are the differential transformations of $u(x)$ and $g(x)$ respectively, then we have the following properties[2]:

If

$$f(x) = \int_{x_0}^{x} g(t)u(t)dt \qquad (7)$$

then

$$F(k) = \sum_{i=0}^{k-1} G(i)\frac{U(k-i-1)}{k}, F(0) = 0 \qquad (8)$$

If

$$f(x) = g(x)\int_{x_0}^{x} u(t)dt \qquad (9)$$

then

$$F(k) = \sum_{i=0}^{k-1} G(i)\frac{U(k-i-1)}{k-1}, F(0) = 0 \qquad (10)$$

4.2 Problem 1

We consider the following linear integral equation :

$$u(x) = x + \int_{0}^{x} tu(t)dt \qquad (11)$$

According to Theorem(4.3.1) equation (7) and (8) and to the operation of differential transformation given in section(3) formula (3.8), we have the following recurrence relation:

$$U(k) = \delta(k-1) + \sum_{t=1}^{k-1} \delta(t-t)\frac{U(k-t-1)}{k} \quad , k \geq 1, U(0) = 0 \qquad (12)$$

Consequently:

when $k = 1$ then $U(1) = 1$, when $k = 2$ then , $U(2) = \frac{1}{2}$,

, when $k = 3$ then $U(3) = \frac{1}{2}$, $,U(4) = \frac{1}{2}, U(5) = \frac{1}{2},\ U(6) = \frac{1}{2}\ ,\ U(7) = \frac{1}{2}\ ,\ U(8) = \frac{1}{2}$

(13)

So the solution is:

$$u(x) = \sum_{k=0}^{\infty} x^k U(k) = 0 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \frac{x^5}{2} + \frac{x^6}{2} + \frac{x^7}{2} + \cdots \qquad (14)$$

or

$$u(x) = \frac{x}{2}\left(\frac{2-x}{1-x}\right) \qquad (15)$$

4.3 Problem 2

Next we consider the following linear integral equation:

$$u(x) = \cos x + \frac{1}{2}\int_{0}^{\frac{\pi}{2}} \sin x\, u(t)dt \qquad (16)$$

According to Theorem(4.3.1) equation (9) and (10)and to the operation of differential transformation given in section(3) formula (3.12) and (3.13), we have the following recurrence relation:

$$U(k) = \frac{1}{k!}\cos(\pi k) + \frac{1}{2}\sum_{n=0}^{k-1}\frac{1}{n!}\sin(\pi k)\frac{U(k-n-1)}{k-n} \qquad (17)$$

with

$$U(0) = 0\,, \qquad k \geq 0 \qquad (18)$$

Thus:

If $k = 1$ then $U(1) = -1$, if $k = 2 \rightarrow U(2) = \frac{1}{2!}$, if $k = 3 \rightarrow U(3) = \frac{-1}{3!}$, if $k = 4 \rightarrow U(4) = \frac{1}{4!}$. Thus:

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k = 0 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= -1 + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad (19)$$

Therefore the required solution is:

$$u(x) = -1 + e^{-x} \qquad (20)$$

4.4 Problem 3

Next we consider the following linear integral equation :

$$u''^{(x)} = e^{-x} - \int_{0}^{x} u^2(t)dt \quad , u(0) = 0, u'(0) = 1 \qquad (21)$$

According to Theorem(4.3.1) equation (9) and (10) and to the operation of differential transformation given in section(3) formula (3.7) , (3.9) and (3.10), we have :

$$(k+1)(k+2)U(k+2) = \left(\frac{(-1)^k}{k!} - \sum_{m=0}^{k-1} u(m)\frac{U(k-m-1)}{k}\right) \qquad (22)$$

Hence, we have the following recurrence relation:

$$U(k+2) = \frac{1}{(k+1)(k+2)}\left(\frac{(-1)^k}{k!} - \sum_{m=0}^{k-1} u(m)\frac{U(k-m-1)}{k}\right) \qquad (23)$$

with the following conditions:

$$U(0) = 0\,, U(1) = 1 \qquad (24)$$

Thus:

If $k = 0$ then $U(2) = \frac{1}{2}$, if $k = 1 \rightarrow U(3) = \frac{-1}{3!}$, if $k = 2 \rightarrow U(4) = \frac{1}{4!}$, if $k = 3 \rightarrow U(5) = \frac{-1}{5!}$ ... (25)

Therefore, the solution is

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k = 0 + x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$
$$= -1 + 2x + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right) \qquad (26)$$

Therefore the required solution is:

$$u(x) = -1 + 2x + e^{-x}$$

4.4 Problem 3

Next we consider the following linear integral equation :

$$u''(x) = 1 + xe^{-x} - \int_0^x e^{-t}u(t)dt \qquad , u(0) = 0, u'(0) = 1 \qquad (27)$$

According to Theorem(4.3.1) equation (9) and (10)and to the operation of differential transformation given in section(3) formula (3.7) , (3.9) and (3.10), we have :

$$(k+1)(k+2)U(k+2) = \left(\delta(k) + \sum_{m=0}^{k} \frac{(-1)^m}{m!}\,\delta(k-1-m) - \sum_{m=0}^{k-1} \frac{(-1)^m}{m!}\frac{U(k-m-1)}{k}\right) \qquad (28)$$

Thus, the recurrence relation is:

$$U(k+2) = \frac{1}{(k+1)(k+2)}\left(\delta(k) + \sum_{m=0}^{k} \frac{(-1)^m}{m!}\,\delta(k-1-m) - \sum_{m=0}^{k-1} \frac{(-1)^m}{m!}\frac{U(k-m-1)}{k}\right) \qquad (29)$$

with the following conditions:

$$U(0) = 0\,, U(1) = 1 \qquad (30)$$

Thus:

If $k = 0$ then $U(2) = \frac{1}{2}$, if $k = 1 \rightarrow U(3) = \frac{1}{3!}$, ...

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k = 0 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= -1 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (31)$$

Therefore the required solution is:

$$u(x) = -1 + e^x$$

V. Conclusion

In this work, we successfully apply the differential transform method to find exact solutions for linear and non-linear integral equations . The present methodreduces the computational difficulties of the other traditional methods and all thecalculations can be made simple manipulations. Several examples were tested by applying the DTM and the results have shown remarkable performance. Therefore, this method can be applied to many Non-linear integral equations without linearization, discretization or perturbation.

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