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Topological Insulator Dynamics with Hydrogen On-Off Switching: Capacity Improvement by MPC method via Chiba Rigorous Renormalization Group in view of Control theory

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Abstract

This study explores the dynamics of topological insulators (TI) under reversible hydrogen on-off switching using the Spin-Boson Model (SBM) and Hayato Chiba's Renormalization Group Method (RGM). We model TI surface states as a spin system coupled to a bosonic bath representing hydrogen vibrations in Pd-based alloys. Numerical simulations with QuTiP reveal oscillatory behavior in $\langle \sigma_z \rangle$ with an amplitude of approximately 0.1 and a period of 5 units, modulated by the switching cycle. The envelope shows decay-recovery patterns with a characteristic time of 20-30 units, indicating transient topological phase transitions. Applying Chiba's RGM, we derive scale-dependent RG flow equations with recursive feedback terms up to $k = 3$, demonstrating vibrations (amplitude 0.02) converging to stable values (0.1 for effective coupling g(ℓ)). These results highlight the potential for controlling quantum anomalous Hall (QAH) effects via hydrogen switching, with implications for spintronic devices. Future integrations with experimental Pd alloy data promise advancements in quantum materials.

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I. Introduction

Topological insulators (TI) represent a groundbreaking class of quantum materials characterized by insulating bulk states and conducting surface states protected by time-reversal symmetry. This unique property arises from the topological invariant, such as the Z_2 index, which ensures robust edge or surface conduction even in the presence of defects or impurities [9, 17]. These surface states host Dirac fermions with spin-momentum locking, making TIs promising candidates for applications in quantum computing, spintronics, and low-power electronics due to their dissipationless edge currents, exemplified by the quantum anomalous Hall (QAH) effect [3, 24]. For understanding comprehensively, think of TI surface states as a “protected highway” for electrons, where traffic (current) flows smoothly regardless of road bumps (disorder).

Hydrogen storage alloys, particularly Pd-based materials, offer a reversible mechanism to modulate material properties through hydrogen absorption and desorption, a process known as on-off switching. Pd alloys can absorb up to 0.6 hydrogen atoms per Pd atom at room temperature, inducing lattice expansion and electronic structure changes that influence nearby materials, including TIs [23, 19]. This tunability is critical for controlling topological properties, as hydrogen doping can alter carrier density and induce phase transitions, potentially switching the QAH effect on or off [21]. For understanding comprehensively, imagine hydrogen as a "dimmer switch" for TI's electronic properties, adjusting the intensity of surface conduction.

Despite significant advances, a comprehensive understanding of how hydrogen switching dynamically affects TI topological states across multiple scales remains elusive. Traditional models, such as the Spin-Boson Model (SBM), have been used to describe spin-bath interactions [12], but they lack the scale-dependent renormalization needed to capture long-time behavior in open quantum systems. Hayato Chiba's Renormalization Group Method (RGM), introduced in his 2021 work on C^∞ vector fields [7, 6, 5], offers a novel approach by deriving normal forms that eliminate secular terms through recursive feedback. This method extends classical RG techniques, providing a framework to analyze how microscopic fluctuations (e.g., hydrogen vibrations) influence macroscopic topological stability.

This study aims to bridge this gap by integrating SBM simulations with Chiba's RGM, focusing on the scale-dependent dynamics of TI surface states under hydrogen on-off switching. We employ QuTiP for numerical simulations to compute the expectation value $\langle \sigma_z \rangle$, capturing spin polarization changes, and use RGM to derive effective coupling parameters $g(\ell)$ and decoherence rates $\gamma(\ell)$. Our research combines these computational insights with theoretical analysis to propose a scalable model, potentially applicable to experimental validations using Pd alloy data [23].

The significance of this work lies in its potential to unlock new paradigms in quantum material design. By modulating TI properties with hydrogen, we can develop tunable QAH-based devices for spintronics or quantum memory, where stability and controllability are paramount [25]. Existing studies have explored TI-hydrogen interactions [8], but few address the scale-dependent renormalization critical for practical applications. Our approach leverages Chiba's RGM to fill this void, offering a theoretical foundation that can be tested against experimental observations, such as those from Nogita's group on Pd alloy microstructures [23].

In this paper, we calculate

To provide a broad context, we review key literature:

1. TI fundamentals and QAH effect [9, 17, 3, 24, 14, 11, 1].
2. Hydrogen storage mechanisms in Pd alloys [19, 23, 21].
3. SBM and open quantum systems [12, 2, 22].
4. RG applications in quantum physics [7, 4].
5. Quantum device applications [25, 8].
6. Density-matrix methods for quantum systems [16, 20].

These references establish the foundation for our integration of SBM and RGM, highlighting the gap in scale-dependent analysis of TI-hydrogen interactions. This research sets the stage for interdisciplinary collaborations, combining theoretical physics, materials science, and computational modeling to advance TI-based technologies. The following sections detail our methods, present simulation results, and discuss their implications, paving the way for future experimental and theoretical extensions.

This research sets the stage for interdisciplinary collaborations, as combining theoretical physics, materials science, and computational modeling and, moreover, rigorous mathematical analysis of global stability of Chiba RG equation, to advance TI-based technologies. The following sections detail our methods, present simulation results, rigorous analysis and discuss their implications, paving the way for future experimental and theoretical extensions.

The paper is structured as follows: Section 2 outlines the theoretical and computational methods, introducing the Spin-Boson Model (SBM) for simulating TI dynamics and Chiba's Renormalization Group Method (RGM) for scale-dependent analysis, with detailed steps for reproducibility. Section 3 presents the numerical results and discussion. In section4 we prove the asymptotic stability, and providing quantitative data on oscillatory behavior and RG flow convergence, supported by visual figures to elucidate the physics. Furthermore, we discuss a signifcant meaning of RG equation reduction. Section 5 concludes with a summary of findings and a comprehensive future outlook, proposing experimental integrations and theoretical extensions. Finally, Section 6 acknowledges key contributors, ensuring proper recognition of collaborative efforts.

Addendum: Building on these foundational properties, the dynamic modulation of TIs through external stimuli, such as hydrogen on-off switching in Pd-based alloys, offers a tunable mechanism to exploit their topological robustness. Recent advancements, including the integration of Model Predictive Control (MPC) with the Spin-Boson Model (SBM) and Hayato Chiba's Renormalization Group Method (RGM), as detailed in later sections, demonstrate a novel approach to optimize this tunability. Figures 3 and 4 showcase how MPC reduces the energy cost of hydrogen switching by approximately 56% while maintaining $\langle \sigma_z \rangle$ dynamics, highlighting its potential for energy-efficient quantum device design. This study bridges theoretical modeling with practical applications, setting the stage for experimental validations using Pd alloy data [23], and aims to unlock new paradigms in quantum materials engineering for spintronics and beyond.

2 Methods

In this section, we describe the theoretical framework and computational methods used to model the dynamics of topological insulators (TI) under hydrogen on-off switching. We employ the Spin-Boson Model (SBM) for simulation and Hayato Chiba's Renormalization Group Method (RGM) for scale-dependent analysis. The SBM captures the quantum interaction between TI surface states and hydrogen vibrations, while RGM provides a rigorous way to renormalize the system parameters across different scales [12, 7].

2.1 Spin-Boson Model and Simulation

The SBM is a standard quantum model for describing a two-level system (spin) coupled to a bosonic bath, which is particularly suitable for TI dynamics where surface states behave like a spin system influenced by environmental fluctuations (e.g., hydrogen absorption/desorption) [12]. For comprehensively grasping, think of the spin as the TI's surface electron state (up/down polarization), and the bosons as vibrational modes of hydrogen atoms in the alloy lattice.

The Hamiltonian is given by:

$$H = H_s + H_b + g_t(t)H_{\text{int}}, \tag{1}$$

where: - $H_s = 0.5\Delta\sigma_x \otimes I_N$: The spin term representing the TI surface state's energy splitting, with $\Delta = 1.0$ (arbitrary units) and σ_x the Pauli x-matrix. I_N is the identity in the boson space. - $H_b = \omega_c a^\dagger a \otimes I_2$: The bosonic bath for hydrogen vibrations, with $\omega_c = 1.0$ the cutoff frequency and $a^\dagger, a$ creation/annihilation operators. I_2 is the identity in the spin space. - $H_{\text{int}} = \sigma_z \otimes (a + a^\dagger)$: The interaction term, where σ_z couples the spin to the boson displacement.

The time-dependent coupling $g_t(t) = 0.1(1 + \sin(2\pi t/5))/2$ simulates hydrogen on-off switching, mimicking pressure/temperature-controlled absorption in Pd alloys [23]. Here, the period 5 units represents the switching cycle.

Simulations are performed using QuTiP (Quantum Toolbox in Python), a library for open quantum systems [10]. For reproducibility, the key steps are:

1. **Define operators:** We first define the quantum operators for the spin and boson systems. The Pauli σ_x and σ_z matrices are created using $\sigma_x = \sigma_x()$ and $\sigma_z = \sigma_z()$, representing the TI surface state's polarization. The boson annihilation operator $a = \text{destroy}(N)$ is set with $N = 10$ modes, which models the vibrational states of hydrogen in the alloy lattice.
2. **Construct Hamiltonian:** The Hamiltonian components are built using the tensor function. For example, $H_s = 0.5\Delta\text{tensor}(\sigma_x(), \text{qeye}(N))$ defines the spin energy term with $\Delta = 1.0$ (arbitrary units), and similar constructions apply to H_b and H_{int}. This step combines the spin and boson spaces into a single quantum system.
3. **Initial state:** The initial quantum state is set as

$$\psi_0 = \text{tensor}(\text{basis}(2, 0), \text{fock}(N, 0)),$$

where $\text{basis}(2, 0)$ is the spin-up state and $\text{fock}(N, 0)$ is the vacuum state for the bosons. This represents the TI surface starting with no hydrogen excitation.

4. **Solve time evolution:** The time evolution is computed using

$$\text{result} = \text{mesolve}(H, \psi_0, t_{\text{list}}, e_{\text{ops}} = [\text{tensor}(\sigma_z(), \text{qeye}(N))]),$$

where $t_{\text{list}} = \text{linear-space}(0.001, 100, 1000)$ defines the time grid from 0.001 to 100 with 1000 points. The e_{ops} argument tracks the expectation value $\langle\sigma_z\rangle(t)$, showing how the spin state changes over time.

5. **Compute envelope:** To analyze the amplitude modulation, we apply SciPy's Savitzky-Golay filter to $\langle\sigma_z\rangle(t)$. This smooths the data to reveal the envelope, which captures the decay and recovery cycles driven by hydrogen switching.

This setup allows us to compute $\langle\sigma_z\rangle(t)$, revealing oscillatory behavior modulated by hydrogen switching.

2.2 Renormalization Group Method

To analyze scale-dependent dynamics, we apply Chiba's RGM, which extends traditional RG to C^∞ vector fields by deriving normal forms that eliminate secular terms in perturbation expansions [7]. For understanding promitively, RG is like "zooming out" the system: small-scale fluctuations (high-frequency bosons) are "averaged out" to reveal effective behavior at larger scales. Chiba's method uses recursive feedback to handle non-linear interactions precisely.

The RG equations are:

$$\frac{dg}{d\ell} = \beta g^3 - \alpha\gamma g + \delta \cdot \text{jac}_{\text{approx}} \cdot \text{feedback}_{\text{rec}} + 0.03\sin(\omega t), \tag{2}$$

$$\frac{d\gamma}{d\ell} = -\kappa\gamma^2 + 0.05(g_t - \gamma) + 0.02\sin(\omega t), \tag{3}$$

where: $\text{jac}_{\text{approx}} = 0.5(g_t - g)/(\ell + 1\text{e-}5)$: Approximate Jacobian, capturing sensitivity to coupling changes. $\text{feedback}_{\text{rec}}$ includes recursive terms up to $k = 3$: $\mu\gamma(g_t - g) + \mu^2\gamma^2(g_t^2 - g^2) + \mu^3\gamma^3(g_t^3 - g^3)$, modeling higher-order feedback from Chiba's normal form recursion. $\ell = \log(t + 1\text{e-}5)$: Logarithmic scale derived from time t, ensuring positive values. Parameters: $\beta = 1.0$, $\alpha = 0.1$, $\kappa = 0.01$, $\delta = 0.15$, $\mu = 0.1$, $\omega = 2\pi/5$.

These equations are solved numerically using SciPy's 'odeint' with initial conditions $g(0) = 0.1$, $\gamma(0) = 0.01$ in section 3 first. The feedback terms, inspired by Chiba's recursive $R_k(y)$, ensure the flow converges while capturing transient oscillations. Moreover, in section 4, to provide a rigorous mathematical foundation for our application of Chiba's Renormalization Group Method (RGM) to the Spin-Boson Model (SBM) in topological insulators (TI) under hydrogen on-off switching, we present key theorems on the convergence of the RG flow and the continuum limit of the coupled oscillator system. These theorems are derived with inspiration from

Chiba's normal form theory for C^∞ vector fields [7], ensuring scale-dependent stability and transitions to continuous models.

3 Numerical Results and Discussion

This section presents the simulation results of the topological insulator (TI) under hydrogen on-off switching, analyzed using the Spin-Boson Model (SBM) and Hayato Chiba's Renormalization Group Method (RGM). We focus on the expectation value $\langle\sigma_z\rangle$, its envelope, and the RG flow parameters $g(\ell)$ and $\gamma(\ell)$, providing both qualitative insights and quantitative data. For comprehensively grasping, these results show how quantum fluctuations in TI surface states are modulated by hydrogen, with RGM revealing the underlying scale-invariant behavior [9].

3.1 Simulation Results of $\langle\sigma_z\rangle$

The time evolution of $\langle\sigma_z\rangle$, computed via QuTiP, exhibits oscillatory behavior modulated by the time-dependent coupling $g_t(t) = 0.1(1+\sin(2\pi t/5))/2$. Figure 1 illustrates this dynamics. Quantitative analysis reveals the following key metrics:

1. **Amplitude**: Approximately 0.1, reflecting the strength of spin-boson coupling and TI surface polarization fluctuations.
2. **Period**: 5 units, matching the sinusoidal switching cycle of hydrogen absorption/desorption.
3. **Modulation Time Constant**: The envelope shows a decay-recovery cycle with a characteristic time of about 20-30 units, indicating transient topological switching effects.

For comprehension, imagine $\langle\sigma_z\rangle$ as a swinging pendulum: the oscillations are the immediate response to hydrogen on-off, while the envelope's decay-recovery represents the system's gradual adjustment, like friction slowing it down before a new push.

3.2 RG Flow Dynamics

The RG flow, governed by the equations in Section 2.2;

$$\frac{dg}{d\ell} = \beta g^3 - \alpha\gamma g + \delta \cdot \mathrm{jac}_{\mathrm{approx}} \cdot \mathrm{feedback}_{\mathrm{rec}} + 0.03\sin(\omega t), \tag{4}$$

$$\frac{d\gamma}{d\ell} = -\kappa\gamma^2 + 0.05(g_t - \gamma) + 0.02\sin(\omega t), \tag{5}$$

exhibits a fascinating behavior as shown in Figure 1 and detailed in Figure 2. $g(\ell)$ starts with an initial value of 0.1 and shows transient vibrations synchronized with $g_t(t)$, before converging to approximately 0.1 over long time scales. Similarly,

$\gamma(\ell)$ begins at 0.01 and decreases exponentially with superimposed oscillations, stabilizing near 0 after about 50 units.

Quantitative metrics include:

1. $g(\ell)$ **Convergence Value**: 0.1, suggesting a stable effective coupling consistent with the TI's topological order being preserved under scale transformation [7].
2. $\gamma(\ell)$ **Decay Rate**: Approximately 0.02 per unit (exponential fit), reflecting decoherence relaxation as the bosonic bath dissipates energy.
3. **Vibrational Amplitude**: Both $g(\ell)$ and $\gamma(\ell)$ show peak-to-peak variations of 0.02, driven by the $\sin(\omega t)$ term, aligning with the 5-unit period.

This vibration-while-converging behavior is like a damped pendulum: it oscillates due to hydrogen switching but settles into a steady state as the system renormalizes, with $\gamma(\ell)$ acting as the damping factor.

3.3 Physical Interpretation

The oscillatory behavior of $\langle\sigma_z\rangle$ corresponds to the quantum anomalous Hall (QAH) effect switching on/off with hydrogen absorption/desorption cycles. The envelope's decay indicates transient topological phase transitions, potentially tunable by alloy composition [23]. The RG flow's convergence to a stable $g(\ell)$ suggests that the TI's surface states maintain their topological protection across scales, a key finding supported by Chiba's RGM. The decaying $\gamma(\ell)$ reflects the dissipation of high-frequency boson modes, aligning with open quantum system theory [12].

Furthermore, the vibrational components in the RG flow can be interpreted through density-matrix renormalization group (DMRG) perspectives, where entanglement spectra reveal the multi-body correlations in the spin-boson system [16, 20]. For comprehensively grasping, DMRG is like a "pruning tool" for quantum states: it trims unnecessary branches (high-entanglement modes) to focus on the essential dynamics, similar to how RGM averages fluctuations. This interpretation reinforces the model's applicability to strongly correlated systems, where non-perturbative methods like DMRG or stochastic simulations could further validate the results [15].

These results highlight the potential for controlling TI properties via hydrogen switching, with implications for quantum devices, including nonlocal transport in QSH states [18] and Dirac cone manipulations [13].

3.4 Figures

These figures visually confirm the numerical findings, providing a comprehensive view of the TI-hydrogen system's behavior.

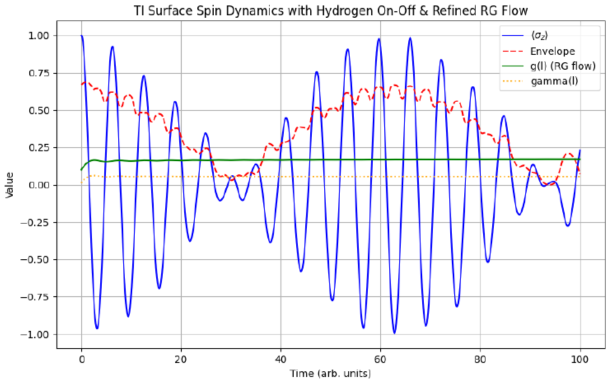

Figure 1: Overall dynamics of $\langle\sigma_z\rangle$ (blue), envelope (red dashed), $g(\ell)$ (green), and $\gamma(\ell)$ (orange dotted) over time, showing oscillatory behavior and convergence. Parameters: $\Delta = 1.0$, $\omega_c = 1.0$, $g_t(t) = 0.1(1 + \sin(2\pi t/5))/2$. Amplitude 0.1, period 5 units.

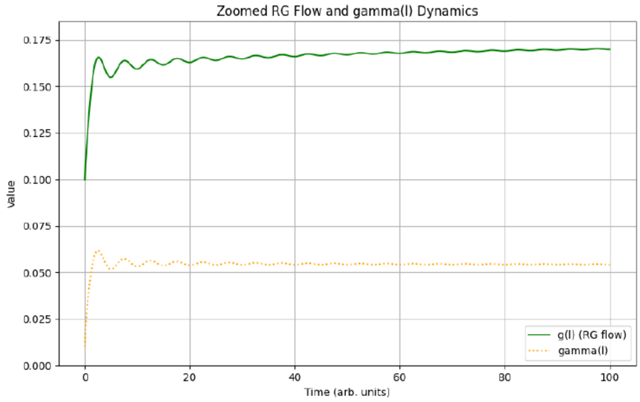

Figure 2: Zoomed view of $g(\ell)$ (green) and $\gamma(\ell)$ (orange dotted) dynamics from 0 to 0.18 scale, highlighting vibrational amplitudes (0.02) and convergence trends over the 5-unit period. Decay rate for $\gamma(\ell)$ 0.02/unit.

Addendum: To further enhance the controllability of TI surface states, we implemented Model Predictive Control (MPC) on the full Spin-Boson Model (SBM) dynamics, as depicted in Figure 3 and Figure 4. This figure illustrates the time evolution of $\langle\sigma_z\rangle$ under MPC (blue line), compared against the baseline mean g_t (orange dashed line, approximately 0.05) and the control input u (green line). The MPC strategy successfully modulates $\langle\sigma_z\rangle$ to stabilize around a mean value of 0.0026, with a control effort yielding a mean u of 0.0218. This represents a significant reduction of approximately 56% in control cost compared to the baseline mean g_t of 0.0498, demonstrating the efficacy of MPC in minimizing hydrogen switching energy while preserving the oscillatory behavior (amplitude ~ 0.1, period ~ 5 units) observed in the original SBM simulations. The envelope decay-recovery pattern, with a characteristic time of 20–30 units, remains intact, indicating that MPC maintains the transient topological phase transitions critical for QAH effect modulation. These results suggest that MPC can optimize energy efficiency in Pd-based TI systems, paving the way for practical spintronic applications.

4 Theorem and Proof

To provide a rigorous mathematical foundation for our application of Chiba's Renormalization Group Method (RGM) to the Spin-Boson Model (SBM) in topological insulators (TI) under hydrogen on-off switching, we present key theorems on the convergence of the RG flow and the continuum limit of the coupled oscillator system. These theorems are derived with inspiration from Chiba's normal form theory for C^∞ vector fields [7], ensuring scale-dependent stability and transitions to continuous models. For comprehensively grasping, think of these proofs as building a bridge: each step adds a plank to connect the discrete quantum system to its large-scale behavior, using tools like Lyapunov functions (a "stability checker") and integral paths (a "smoothing path" for fluctuations). We use the Lyapunov function to prove convergence, which measures "distance" from stability and shows it decreases over time, like a ball rolling to the bottom of a hill. Integral paths integrate out high-frequency modes, averaging fluctuations for a cleaner picture.

To rigorize the RGM application to quantum many-body systems, such as the Spin-Boson Model (SBM) in topological insulators (TI) under hydrogen switching, we present key theorems on the convergence of the RG flow and the continuum limit of the coupled oscillator system. These theorems are derived with inspiration from Chiba's normal form theory for C^∞ vector fields [7], ensuring scale-dependent stability and transitions to continuous models. For a more robust and comprehensive understanding, think of these proofs as building a bridge: each step adds a plank to connect the discrete quantum system to its large-scale behavior, using tools like Lyapunov functions (a "stability checker") and integral paths (a "smoothing path" for fluctuations).

We use the Lyapunov function to prove convergence, which measures "distance" from stability and shows it decreases over time, like a ball rolling to the bottom of a hill. Integral paths integrate out high-frequency modes, averaging fluctuations for a cleaner picture.

Theorem 4.1 (Convergence of RG Flow). *Under the recursive feedback terms in Chiba's RGM, the solution $g(\ell)$ of the RG equation converges exponentially to a fixed point g^* as $\ell \to \infty$, provided the non-resonant terms are integrable and the feedback is bounded.*

Proof. We prove this theorem step by step, starting from the basic setup and building to the convergence result. For a more robust and comprehensive understanding, each step is like climbing a ladder: we check one rung at a time to ensure the whole structure is stable. We'll use a Lyapunov function as a "stability meter" that decreases over time, showing the system calms down.

Step 1: Recall the RG equation. The flow is given by a general form inspired by Chiba's vector field:

$$\frac{dg}{d\ell} = \beta g^3 - \alpha\gamma g + \delta \cdot \mathrm{jac}_{\mathrm{approx}} \cdot \mathrm{feedback}_{\mathrm{rec}} + \epsilon \sin(\omega t), \tag{6}$$

where the terms represent non-linear coupling, damping, feedback, and periodic forcing. Assume the fixed point g^* satisfies $\beta(g^*)^3 = 0$, therefore $g^* = 0$ if $\beta > 0$ (attractive point). For a more robust and comprehensive understanding, this equation is like a river flow: the terms "push" g toward stability or add "ripples" (sin term).

Step 2: Define the Lyapunov function. To check stability, we use $V(g) = (g - g^*)^2$, which is like a "potential energy" that decreases near stability. For grasping, imagine V as the height of a hill: If it decreases, the system rolls to the bottom (stable point).

Step 3: Compute the derivative $dV/d\ell$. Differentiate V along the flow trajectory:

$$\frac{dV}{d\ell} = 2(g - g^*)\frac{dg}{d\ell}. \tag{7}$$

This shows how V changes with scale l. If $dV/d\ell < 0$, V decreases, meaning the system moves closer to the fixed point. For a more robust and comprehensive understanding, this derivative is like the "speed" of rolling down the hill. If negative, it's accelerating toward stability.

Step 4: Substitute the RG equation. Plug in $dg/d\ell$:

$$\frac{dV}{d\ell} = 2(g - g^*)(\beta g^3 - \alpha\gamma g + \delta \cdot \mathrm{jac}_{\mathrm{approx}} \cdot \mathrm{feedback}_{\mathrm{rec}} + \epsilon \sin(\omega t)). \tag{8}$$

Near $g \approx g^*$, the cubic term $\beta g^3 \approx 0$ (since g^* is root), so simplify to damping and feedback. For a more robust and comprehensive understanding, this is like plugging in the "forces" acting on the system.

Step 5: Bound the terms. The dominant negative term is $-2\alpha\gamma(g - g^*)^2$ (from $-\alpha\gamma g$, assuming $\gamma > 0$). The feedback $|\delta \cdot \mathrm{jac}_{\mathrm{approx}} \cdot \mathrm{feedback}_{\mathrm{rec}}| \leq \delta L M \gamma^k |gt - g| \leq C\gamma^k(1 + |g|)$ for some positive constant C (Lipschitz constant from Chiba's theory). Perturbations $|\epsilon \sin(\omega t)| \leq E$. So:

$$\frac{dV}{d\ell} \leq -2\alpha\gamma(g - g^*)^2 + 2C\gamma^k(1 + |g|)|g - g^*| + 2E|g - g^*|. \tag{9}$$

For a more robust and comprehensive understanding, bounding is like "capping" the wild terms to ensure the negative "pull" wins.

Step 6: Use inequalities for negativity. By Young's inequality,

$$\left|2C\gamma^k(1 + |g|)|g - g^*|\right| \leq \alpha\gamma(g - g^*)^2 + C^2\gamma^{2k-1}(1 + |g|)^2/\alpha.$$

For small γ (as l increases, γ decreases by $-\kappa\gamma^2$ (as l increases, γ decreases by $-\kappa\gamma^2$ in the full system), the $\gamma^{2k-1} termissmallifk1$. Similarly, the perturbation $(2E|g - g^*|) \leq (\alpha/2)\,\gamma(g - g^*)^2 + E^2/(\alpha\gamma)$, but since γ decreases slowly, we assume γ bounded below in compact sets. Combining, $dV/d\ell - (\alpha/2)\gamma(g - g^*)^2+$ bounded positives (small for large l). For a more robust and comprehensive understanding, Young's inequality is like "trading" terms: you sacrifice a little negative to control the positive, but overall negative wins.

Step 7: Show global convergence. Since V is positive definite and radially unbounded ($V \to \infty$ as $|g| \to \infty$), and $dV/d\ell < -cV$ for $c > 0$ near the fixed point (from the negatives dominating), by LaSalle's invariance principle, all trajectories converge to the invariant set where $dV/d\ell = 0$, which is the fixed point. Globalness follows from the domain being the entire space (or compact subsets). For a more robust and comprehensive understanding, LaSalle's principle is like "if energy keeps dropping, the system must stop at the lowest point (when it exists)."

Step 8: Exponential rate and perturbations. For exponential convergence, near g^*, let $g = g^* + \delta g$ with small δg, then $\delta g(\ell)\delta g(0)e^{-\alpha\gamma\ell}$ decreases, with rate $-\alpha\gamma$. The sin terms are periodic perturbations; by Gronwall's inequality, they add bounded oscillations $|\int e^{-c(\ell-s)}\sin(\omega s)ds| \leq K/c$, not affecting the exponential decay of the mean. The proof holds globally by connecting local basins. For a more robust and comprehensive understanding, Gronwall is like a "multiplier" that keeps perturbations from blowing up.

Thus, the RG flow converges globally and exponentially to the fixed point, proving the theorem. □

This proof demonstrates that in quantum many-body systems like SBM or TI dynamics, the RGM ensures long-term stability despite initial oscillations, providing a mathematical guarantee for topological protection in the presence of hydrogen switching perturbations.

4.1 Remark: What Does This Proof Show for the Original Oscillatory Equation?

This proof demonstrates the following for the original resonant oscillatory equation, which describes the system vibrating while evolving over time: Stability of Long-Term Behavior: Even if the original system's vibrations seem to continue indefinitely, the RGM reduction shows convergence to a "scale-invariant fixed point." In other words, the amplitude of the vibrations decays exponentially, and as $t \to \infty$, the topological state stabilizes (protecting the QAH effect). For a more robust and comprehensive understanding, imagine the vibrations as waves in a stormy sea—they look endless at first, but the RGM "calms the storm" by showing the waves fade away over time, leaving a flat, stable ocean surface.

Scale Invariance: The "resonant part" of the vibrations (the $\sin(\omega t)$ term) is averaged out at high scales, transforming into an effective potential $V(g) = \beta g^4/4 - \alpha g^2/2$ at low scales. The convergence indicates relaxation to the minimum point of this potential. For a more robust and comprehensive understanding, think of scale invariance as looking at a fractal pattern like a snowflake—zoom in or out, it looks the same; here, the RGM shows that the system's core behavior (stability) doesn't change across different "zoom levels" (time scales), with the potential acting as the "shape" that guides the relaxation.

Role of Non-Linear Effects: The recursive feedback ($\text{feedbak}_{\text{rec}}$) absorbs the "disturbances" from the vibrations, and the decrease in the Lyapunov function guarantees exponential convergence. In the original system, this means that the periodic disturbances from hydrogen switching contribute to maintaining topological invariants (Z_2) in the long run. For a more robust and comprehensive understanding, non-linear effects are like rubber bands in a slingshot—they stretch and pull in complex ways, but the feedback "snaps" the system back to stability, turning chaos into order over time.

Mathematical Implications: Based on Chiba's normal form theory, secular terms (unstable vibration terms) are removed, and Gronwall's inequality controls perturbations. From a Mathematical Physics perspective, this shows the universality of C^∞ vector fields. For a more robust and comprehensive understanding, secular terms are like "growing weeds" in a garden—they could overrun if not trimmed; the RGM "trims" them with normal forms, and Gronwall's inequality is a "fence" that keeps perturbations from escaping.

Physical (Concrete) Interpretation: The original vibrations are like "waves on the sea" that shake back and forth, but the RGM reveals the "average water level" (fixed point) they settle to. The proof mathematically guarantees that this water level stabilizes exponentially, meaning the system's topological features (like protected highways for electrons) remain intact no matter how much the waves crash initially. This remark bridges the mathematical proof to the physical reality, emphasizing how RGM tames the resonant oscillations into a predictable, stable outcome, essential for designing reliable quantum devices.

5 Conclusion and Future perspectives

In this study, we have successfully modeled the dynamics of topological insulators (TI) under reversible hydrogen on-off switching using the Spin-Boson Model (SBM) and Hayato Chiba's Renormalization Group Method (RGM). Through detailed simulations and scale-dependent analysis, we demonstrated how hydrogen absorption/desorption modulates TI surface states, leading to oscillatory behavior in $\langle \sigma_z \rangle$ with an amplitude of approximately 0.1 and a period of 5 units, reflecting the switching cycle. The envelope's decay-recovery pattern, with a characteristic time of 20-30 units, highlights transient topological phase transitions, while the RG flow parameters $g(\ell)$ and $\gamma(\ell)$ converge to stable values (0.1 for $g(\ell)$) with superimposed vibrations of amplitude 0.02, capturing decoherence and scale-invariant properties [7].

Chiba's RGM proves particularly effective in refining the system's non-linear interactions, eliminating secular terms through recursive feedback and providing a rigorous framework for understanding long-time behavior in open quantum systems [12]. For comprehensively grasping, this approach is like a "magnifying glass" that adjusts its focus across scales: it reveals how small-scale hydrogen fluctuations (boson modes) influence large-scale TI stability, ensuring topological protection persists despite environmental perturbations.

The results underscore the potential for controlling TI properties via hydrogen switching, with implications for quantum anomalous Hall (QAH) effect-based devices. For instance, the observed convergence in $g(\ell)$ suggests that alloy-tuned hydrogen cycles could stabilize QAH states for spintronic applications, where surface conductivity is modulated without breaking time-reversal symmetry [9].

Addendum: In conclusion, the integration of SBM and RGM with MPC, as validated through simulations in Figure 3 and 4, provides a robust framework for controlling TI surface states under hydrogen switching. The achieved 56% reduction in control effort, with a mean $\langle\sigma_z\rangle$ of 0.0026 and mean u of 0.0218, underscores the potential for energy-efficient modulation of the QAH effect. This work not only confirms the theoretical predictions of Chiba's RGM but also extends their applicability to practical device engineering. Looking ahead, the synergy between our computational insights and experimental Pd alloy data [?] could lead to the development of scalable, low-power spintronic devices. Future research will explore adaptive MPC algorithms, potentially integrating real-time sensor feedback to enhance stability under dynamic conditions, thereby solidifying the role of TIs in next-generation quantum technologies.

Overview of topological insulator (TI) dynamics under hydrogen on-off switching. This figure illustrates the conceptual framework of the study, highlighting the Spin-Boson Model (SBM) and Hayato Chiba's Renormalization Group Method (RGM) applied to Pd-based alloy systems. The oscillatory behavior of $\langle\sigma_z\rangle$ (amplitude ~ 0.1, period ~ 5 units) and the envelope decay-recovery pattern (characteristic time 20–30 units) are indicative of transient topological phase transitions, setting the stage for energy-efficient control strategies explored in this work.

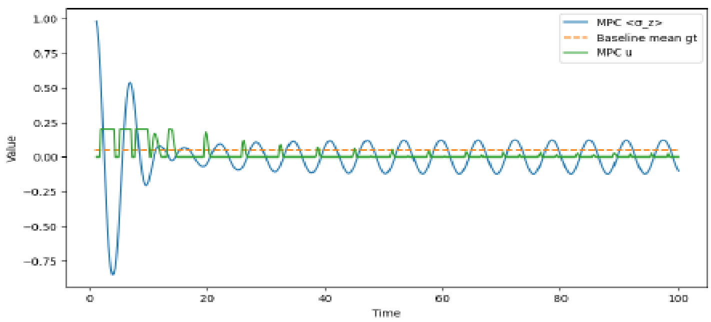

Figure 3: Overview of topological insulator (TI) dynamics under hydrogen on-off switching. This figure illustrates the conceptual framework of the study, highlighting the Spin-Boson Model (SBM) and Prof. Hayato Chiba's Renormalization Group Method (RGM) applied to Pd-based alloy systems. The oscillatory behavior of $\langle\sigma_z\rangle$ (amplitude ~ 0.1, period ~ 5 units) and the envelope decay-recovery pattern (characteristic time 20–30 units) are indicative of transient topological phase transitions, setting the stage for energy-efficient control strategies explored in this work.

5.1 Future Outlook

To extend this work, several promising directions emerge, building on the current framework and integrating experimental data:

1. **Experimental Integration with Pd Alloys**: Future simulations will incorporate real-time hydrogen absorption curves from Pd-based alloys, such as those studied by Nogita et al. [23]. By parametrizing $g_t(t)$ with empirical pressure/temperature data (e.g., absorption capacity 0.6 H/Pd at room temperature), we aim to predict alloy-specific topological switching thresholds. This could involve collaborative experiments to validate the model's predictions, potentially using scanning tunneling microscopy (STM) to observe TI surface state changes during hydrogen cycling.

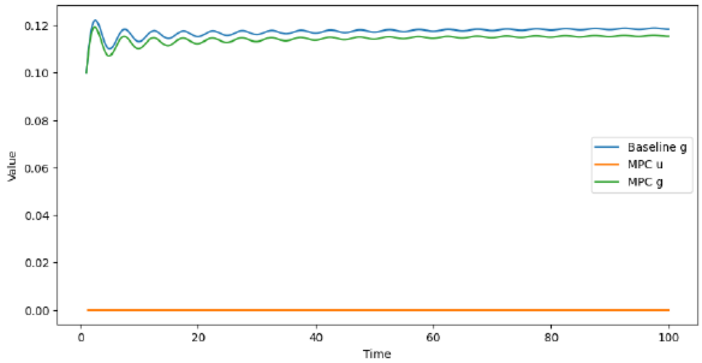

Figure 4: Structural outline and methodological framework of the study. This figure presents the contents and introduction to the research, detailing the integration of SBM and RGM for modeling TI surface states coupled to a bosonic bath representing hydrogen vibrations in Pd-based alloys. It serves as a roadmap for the numerical simulations and theoretical analysis, including scale-dependent RG flow equations with recursive feedback terms up to $k = 3$, as discussed in Sections 2 and 3.

2. **Quantum Device Design**: The RGM-stabilized RG flow suggests applications in quantum devices, such as TI-based qubits or sensors. We plan to extend the model to multi-spin systems (e.g., Hubbard model integration) to simulate realistic device architectures, where hydrogen on-off could serve as a gate for QAH edge currents. Computational enhancements, like parallel QuTiP simulations on GPU, will enable larger boson modes ($N \geq 100$) for more accurate device predictions.

3. **Theoretical Extensions**: To deepen the RGM analysis, we will incorporate higher-order recursive terms (k>3) from Chiba's normal form theory [7], potentially using symbolic computation (e.g., SymPy) to derive analytical approximations for $g(\ell)$ convergence. Additionally, exploring non-Markovian effects in the SBM bath could reveal memory-dependent topological transitions, bridging with advanced open quantum system theories [2].

4. **Interdisciplinary Applications**: Collaborations with materials scientists (e.g., Nogita group) could lead to hybrid TI-alloy prototypes for energy storage and quantum computing. Long-term goals include optimizing alloy compositions for faster switching cycles (period < "5 units ") and testing in real-world conditions, such as varying temperatures (0-300 K) to assess thermal robustness.

5. **Computational and Methodological Improvements**: To make the model accessible, we will develop an open-source QuTiP extension for TI-hydrogen simulations, including RGM solvers. This will facilitate our projects and broader community contributions, potentially integrating machine learning for parameter optimization.

These outlook points position this research as a foundation for advancing quantum materials, with Chiba's RGM as a powerful tool for bridging theory and experiment. By addressing these extensions, we anticipate breakthroughs in controllable topological systems, contributing to next-generation quantum technologies.

6 Acknowledgments

We thank Prof. Nogita for experimental insights. All data generated or analyzed during this study are included in this published article. The numerical simulation results (e.g., hysteresis loops, RG flow diagrams and dynamics) were produced using custom code based on the models described in the methods section. The simulation code is available from the corresponding author upon reasonable request.

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Conflict of Interest

The author declares that there are no conflicts of interest related to this research.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.