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Common Fixed Point Theorems for Weakly Compatible Mappings of Integral Type Contraction in 2-Metric Spaces

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ABSTRACT
Branciari [3] introduced the notion of contractions of integral type and proved fixed point theorem for this class of mappings. Further results on this class of mappings were obtained by Rhoades [15] and many others. In this paper we prove a common fixed point theoremof integral type contraction forweakly compatible mappings in complete 2–metric spaces. In this process, many known results are enriched and improved.

KEY WORDS: *complete 2–metric spaces, common fixed points, weakly compatiblemappings.*
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I. INTRODUCTION AND PRELIMINARIES:

Fixed point theory is a rich, interesting and exciting branch of mathematics. It is relatively young but fully developed area of research. Study of the existence of fixed points falls within several domains such as functional analysis, operator theory, general topology. Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as topology, mathematical economics, game theory, approximation theory and initial and boundary value problems in ordinary and partial differential equations. Moreover, recently, the usefulness of this concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major tool in the arsenal of mathematics.

Polish mathematician Banach [2] proved a theorem, which ensures under appropriate conditions, the existence and uniqueness of a fixed point. This result is popularly known as "Banach fixed point theorem" or the "Banach Contraction Principle". It states that a contraction mapping of a complete metric space into itself has a unique fixed point. It is the simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map, it produces approximations of any required accuracy. Due to its applications in various disciplines of mathematics and mathematical sciences, the Banach contraction principle has been extensively studied and generalized on many settings and fixed point theorems have been established.

Jungck [9] generalized the Banach contraction principle by introducing a contractive condition for a pair of commuting self mapping son metric space and pointed out the potential of commuting mappings for generalizing fixed point theorems in metric spaces. Jungck's [10] results have been further generalized.

Sessa [16], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa [16] introduced the notion of weak commutativity. Motivated by Sessa [16], Jungck [11] defined the concept of compatibility of two mappings, which includes weakly commuting mappings as a proper subclass.Jungck and Rhoades [13] introduced the notion of weakly compatible (coincidentally commuting) mappings and showed that compatible mappings are weakly compatible but not conversely. Many interesting fixed point theorems for weakly compatible maps satisfying contractive type conditions have been obtained by various authors.

The concept of **2-metric spaces** has been investigated initially by Gahler [5]. This concept was subsequently enhanced by Gahler ([6], [7]), White [19] and several others. On the other hand Guay and Singh [8], Sharma and Yuel [17], Ciric [4]and a number of other authors have studied the aspects of fixed point theory

in the setting of 2-metric space. They have been motivated by various concepts already known for metric spaces and have thus introduced analogous of various concepts in the frame-work of the 2-metric space.
Following definitions are given by Gahler [5,6] and White [19]:

Definition1.1: Let X be a set consisting of atleast three points. A **2–metric** on X is a real valued function$d: X \times X \times X \to R^+$, which satisfies the following conditions:

(i) To each pair of distinct pointsx, y in X, there exits a point z in X such that $d(x, y, z) \neq 0$,

(ii) $d(x, y, z) = 0$, when at least two ofx, y, z are equal;

(iii) $d(x, y, z) = d(y, z, x) = d(x, z, y)$for all x, z, yin X,

(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.

The pair (*X*, *d*) is called a **2–metric space**.

Definition 1.2:A sequence$\{x_n\}$ in a 2–metric space(X, d) is said to be **convergent** with limit x in X if$\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X.

Definition 1.3:A sequence $\{x_n\}$ in a 2–metric space (X, d) is said to be a **Cauchy sequence** if $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$ for all a in X.

Definition1.4:A 2–metric space (X, d) is said to be **complete** if every Cauchy sequence in X is convergent.

Sessa [16] introduced the notion of weakly commuting mapping as follows–

Definition1.5: Let (X, d) be a 2–metric space and$S$, Tbe self mappings of X. Then $(S, T)$is said to be **weakly commuting** pair if

$$d(STx, TSx, a) \leq d(Sx, Tx, a) \text{ for all} x, a \text{ in } X.$$

Jungck [11] proposed the concept of compatibility is described as:

Definition1.6: Let S and T be self mappings of a 2–metric space (X, d). Then the pair (S, T) is called **compatible** if,

$$\lim_{n\to\infty} d(STx_n, TSx_n, a) = 0 \text{for all } a \text{ in } X,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

Definition 1.7: Let S and T be two self maps on a set X, if $Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T.

Definition 1.8: Let S and T be two self mappings on 2–metric space (X, d), Then S and T are said to be weakly compatible if they commute at their coincidence points i.e. if, for every $x \in X$, holds $STx = TSx$ wherever $Sx = Tx$.

Lemma 1.9: Let $\emptyset \in \varphi$ where $\varphi = \{\emptyset : \emptyset : R^+ \to R^+$ satisfies that $\emptyset$ is Lebesgue integrable, summable on each compact subset of R^+ and $\int_0^{\epsilon} \emptyset(t)dt) > 0$ for each $\epsilon > 0\}$ and $\{r_n\}_{n\in N}$ be a non–negative sequence, then $\lim_{n\to\infty} \int_0^{r_n} \emptyset(t)dt) = 0$iff $\lim_{n\to\infty} r_n = 0$.

II. MAIN RESULTS

In this section, we introduced more general contractive mappings of integral type and establish the existence and uniqueness of a common fixed point for these contractive mappings of integral type in 2–metric space by using weak compatibility. Our result extend and improve several known results.

Theorem2.1:Let (X, d) be a complete 2–metric space. Let A, B, S, T, I and J be six self mappings on X satisfying the following conditions

(i) $J(X) \subseteq AB(X), I(X) \subseteq ST(X)$ … (1)

(ii) $\int_0^{d(Ix,Jy,a)} \emptyset(t)dt \leq \psi(max\{\int_0^{m_i(x,y,a)} \emptyset(t)dt : 1 \leq i \leq 4\})$ … (2)

$\forall\, x, y, a \in X$ where $\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of R^+ and $\int_0^{\epsilon} \emptyset(t)dt > 0$ for each $\epsilon > 0$. $\psi: R^+ \to R^+$ is non–negative and non–decreasing on R^+, $\psi(t) < t$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ and

$$m_1(x, y, a) = d(STy, Jy, a)\left[\frac{1+d(ABx,Ix,a)}{1+d(ABx,STy,a)}\right]$$

$$m_2(x, y, a) = d(ABx, Ix, a)\left[\frac{1+d(STy,Jy,a)}{1+d(ABx,STy,a)}\right]$$

$$m_3(x, y, a) = \frac{d(Ix,STy,a)\, d(Jy,ABx,a)}{1+d(ABx,STy,a)} \quad \ldots (3)$$

$$m_4(x, y, a) = max\{d(ABx, STy, a), d(ABx, Ix, a), d(STy, Jy, a), \frac{1}{2}[d(Ix, STy, a) + d(Jy, ABx, a)]\}$$

(iii) One of $AB(X), ST(X), I(X)$ and $J(X)$ is a complete subset of X then

(a) (AB, I) has a coincidence point

(b) (ST, J) has a coincidence point … (4)

Moreover, if the pairs (AB, I) and (ST, J) are coincidentally commuting (weakly compatible), then AB, ST, I and J have a unique common fixed point.
Furthermore, if the pairs $(A, B), (A, I), (B, I), (S, T), (S, J)$ and (T, J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.
Proof:
Let $x_0 \in X$ be an arbitrary point. Since $I(X) \subseteq ST(X)$, we can choose a point x_1 in X such that $Ix_0 = STx_1$. Again $J(X) \subseteq AB(X)$, we can choose a point x_2 in X such that $Jx_1 = ABx_2$. Continuing this process, we can define a sequence $\{z_n\}_{n\in N}$ and $\{x_n\}_{n\in N_0}$ in X satisfying

$$z_{2n+1} = Ix_{2n} = STx_{2n+1}$$

$z_{2n+2} = Jx_{2n+1} = ABx_{2n+2} \forall\, n \in N_0$.
Suppose that $z_n \neq z_{n+1}$ for all $n \in N$.
Now, we claim that $d(z_{2n}, z_{2n+1}, a) \leq d(z_{2n-1}, z_{2n}, a) \forall\, n \in N$. Suppose that $d(z_{2n}, z_{2n+1}, a) > d(z_{2n-1}, z_{2n}, a)$ for some $n \in N$. Then

$$m_1(x_{2n}, x_{2n-1}, a) = d(STx_{2n-1}, Jx_{2n-1}, a)\left[\frac{1+d(ABx_{2n}, Ix_{2n}, a)}{1+d(ABx_{2n}, STx_{2n-1}, a)}\right]$$
$$= d(z_{2n-1}, z_{2n}, a)\left[\frac{1+d(z_{2n}, z_{2n+1}, a)}{1+d(z_{2n}, z_{2n-1}, a)}\right]$$
$$< (z_{2n}, z_{2n+1}, a)$$
$$m_2(x_{2n}, x_{2n-1}, a) = d(ABx_{2n}, Ix_{2n}, a)\left[\frac{1+d(STx_{2n-1}, Jx_{2n-1}, a)}{1+d(ABx_{2n}, STx_{2n-1}, a)}\right]$$
$$= d(z_{2n}, z_{2n+1}, a)\left[\frac{1+d(z_{2n-1}, z_{2n}, a)}{1+d(z_{2n}, z_{2n-1}, a)}\right]$$
$$= d(z_{2n}, z_{2n+1}, a)$$
$$m_3(x_{2n}, x_{2n-1}, a) = \frac{d(Ix_{2n}, STx_{2n-1}, a)\, d(Jx_{2n-1}, ABx_{2n}, a)}{1+d(ABx_{2n}, STx_{2n-1}, a)}$$
$$= \frac{d(z_{2n+1}, z_{2n-1}, a)\, d(z_{2n}, z_{2n}, a)}{1+d(z_{2n}, z_{2n-1}, a)} = 0$$
$$m_4(x_{2n}, x_{2n-1}, a) = max\{d(ABx_{2n}, STx_{2n-1}, a), d(ABx_{2n}, Ix_{2n}, a),$$
$$d(STx_{2n-1}, Jx_{2n-1}, a), \tfrac{1}{2}[d(Ix_{2n}, STx_{2n-1}, a)$$
$$+d(Jx_{2n-1}, ABx_{2n}, a)]\}$$
$$= max\{d(z_{2n}, z_{2n-1}, a), d(z_{2n}, z_{2n+1}, a),$$
$$d(z_{2n-1}, z_{2n}, a), \tfrac{1}{2}[d(z_{2n+1}, z_{2n-1}, a)$$
$$+d(z_{2n}, z_{2n}, a)]\} \ldots (5)$$

∴ $$\tfrac{1}{2}d(z_{2n+1}, z_{2n-1}, a) \leq \tfrac{1}{2}[d(z_{2n+1}, z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n-1}, a)$$
$$+d(z_{2n+1}, z_{2n}, a)]$$

Now, we have

$$m_1(x_{2n}, x_{2n-1}, z_{2n-1}) = d(STx_{2n-1}, Jx_{2n-1}, z_{2n-1})\left[\frac{1 + d(ABx_{2n}, Ix_{2n}, z_{2n-1})}{1 + d(ABx_{2n}, STx_{2n-1}, z_{2n-1})}\right]$$
$$= d(z_{2n-1}, z_{2n}, z_{2n-1})\left[\frac{1+d(z_{2n}, z_{2n+1}, z_{2n-1})}{1+d(z_{2n}, z_{2n-1}, z_{2n-1})}\right] = 0$$
$$m_2(x_{2n}, x_{2n-1}, z_{2n-1}) = d(ABx_{2n}, Ix_{2n}, z_{2n-1})\left[\frac{1 + d(STx_{2n-1}, Jx_{2n-1}, z_{2n-1})}{1 + d(ABx_{2n}, STx_{2n-1}, z_{2n-1})}\right]$$
$$= d(z_{2n}, z_{2n+1}, z_{2n-1})\left[\frac{1+d(z_{2n-1}, z_{2n}, z_{2n-1})}{1+d(z_{2n}, z_{2n-1}, z_{2n-1})}\right]$$
$$= d(z_{2n}, z_{2n+1}, z_{2n-1})$$
$$m_3(x_{2n}, x_{2n-1}, z_{2n-1}) = \frac{d(Ix_{2n}, STx_{2n-1}, z_{2n-1})\, d(Jx_{2n-1}, ABx_{2n}, z_{2n-1})}{1 + d(ABx_{2n}, STx_{2n-1}, z_{2n-1})}$$
$$= \frac{d(z_{2n+1}, z_{2n-1}, z_{2n-1})\, d(z_{2n}, z_{2n}, z_{2n-1})}{1+d(z_{2n}, z_{2n-1}, z_{2n-1})} = 0$$
$$m_4(x_{2n}, x_{2n-1}, z_{2n-1}) = max\{d(ABx_{2n}, STx_{2n-1}, z_{2n-1}), d(ABx_{2n}, Ix_{2n}, z_{2n-1}),$$
$$d(STx_{2n-1}, Jx_{2n-1}, z_{2n-1}), \tfrac{1}{2}[d(Ix_{2n}, STx_{2n-1}, z_{2n-1})$$
$$+d(Jx_{2n-1}, ABx_{2n}, z_{2n-1})]\}$$
$$= max\{d(z_{2n}, z_{2n-1}, z_{2n-1}), d(z_{2n}, z_{2n+1}, z_{2n-1}),$$
$$d(z_{2n-1}, z_{2n}, z_{2n-1}), \tfrac{1}{2}[d(z_{2n+1}, z_{2n-1}, z_{2n-1})$$
$$+d(z_{2n}, z_{2n}, z_{2n-1})]\}$$
$$= d(z_{2n}, z_{2n+1}, z_{2n-1})$$

and

$$\int_0^{d(z_{2n+1}, z_{2n}, z_{2n-1})} \emptyset(t)dt = \int_0^{d(Ix_{2n}, Jx_{2n-1}, z_{2n-1})} \emptyset(t)dt$$

$\leq \psi(max\,\{\int_0^{m_i(x_{2n},x_{2n-1},z_{2n-1})} \emptyset(t)dt : 1 \leq i \leq 4\})$
$= \psi(max\,\{0, \int_0^{d(z_{2n},z_{2n+1},z_{2n-1})} \emptyset(t)dt\,, 0, \int_0^{d(z_{2n},z_{2n+1},z_{2n-1})} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z_{2n},z_{2n+1},z_{2n-1})} \emptyset(t)dt) < \int_0^{d(z_{2n},z_{2n+1},z_{2n-1})} \emptyset(t)dt$

which is a contradiction.

Hence $\int_0^{d(z_{2n+1},z_{2n},z_{2n-1})} \emptyset(t)dt = 0$

or, $(z_{2n+1}, z_{2n}, z_{2n-1}) = 0$ [by def, of $\emptyset$] … (6)

Hence,

$\frac{1}{2} d(z_{2n+1}, z_{2n-1}, a) \leq \frac{1}{2}[d(z_{2n}, z_{2n-1}, a) + d(z_{2n+1}, z_{2n}, a)$ [by (6)]
$= max\{d(z_{2n}, z_{2n-1}, a), d(z_{2n}, z_{2n+1}, a)\}$

∴ $m_4(x_{2n}, x_{2n-1}, a) = d(z_{2n}, z_{2n+1}, a)$

Now,

$\int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt) = \int_0^{d(Ix_{2n},Jx_{2n-1},a)} \emptyset(t)dt$
$\leq \psi(max\,\{\int_0^{m_i(x_{2n},x_{2n-1},a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$= \psi(max\,\{\int_0^{d(z_{2n-1},z_{2n},a)\left[\frac{1+d(z_{2n},z_{2n+1},a)}{1+d(z_{2n},z_{2n-1},a)}\right]} \emptyset(t)dt,$
$\int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt, 0, \int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt)$
$< \int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt$

which is a contradiction.

Hence, $d(z_{2n}, z_{2n+1}, a) \leq d(z_{2n-1}, z_{2n}, a)$ for each $n \in N$.

Similarly, $d(z_{2n+1}, z_{2n+2}, a) \leq d(z_{2n}, z_{2n+1}, a) \forall\, n \in N$.

Consequently, $\{d(z_n, z_{n+1}, a)\}_{n\in N}$ or $\{d_n\}_{n\in N}$ is a non–decreasing positive sequence which means that there exists a constant $r \geq 0$ with $\lim_{n\to\infty} d_n = r$ where $d_n = d(z_n, z_{n+1}, a)$ … (7)

Suppose that $r > 0$. Making use of (2), (3), (4), (7) and Lemma(1.9), we get

$\int_0^r \emptyset(t)dt = \lim_{n\to\infty} Sup \int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt$
$= \lim_{n\to\infty} Sup \int_0^{d(z_{2n+1},z_{2n},a)} \emptyset(t)dt$
$= \lim_{n\to\infty} Sup \int_0^{d(Ix_{2n},Jx_{2n-1},a)} \emptyset(t)dt$
$\leq \lim_{n\to\infty} Sup\, \psi(max\,\{\int_0^{m_i(x_{2n},x_{2n-1},a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$\leq \psi(\lim_{n\to\infty} Sup\,(max\,\{\int_0^{d(z_{2n-1},z_{2n},a)\left[\frac{1+d(z_{2n},z_{2n+1},a)}{1+d(z_{2n+1},z_{2n},a)}\right]} \emptyset(t)dt,$
$\int_0^{d(z_{2n},z_{2n+1},a)} \emptyset(t)dt, 0, \int_0^{max\{d(z_{2n-1},z_{2n},a),d(z_{2n},z_{2n+1},a)\}} \emptyset(t)dt\}))$
$= \psi(max\,\{\int_0^r \emptyset(t)dt\,, \int_0^r \emptyset(t)dt, 0, \int_0^r \emptyset(t)dt\})$
$= \psi(\int_0^r \emptyset(t)dt) < \int_0^r \emptyset(t)dt$

which is a contradiction.

Hence, $r = 0$ i.e. $\lim_{n\to\infty} d_n = 0$ where $d_n = d(z_n, z_{n+1}, a)$

$\Rightarrow \lim_{n\to\infty} \int_0^{d(z_n,z_{n+1},a)} \emptyset(t)dt = 0$ [By Lemma (1.9)] … (8)

Using property (iv) of 2–metric space

$d(z_n, z_{n+2}, a) \leq d(z_n, z_{n+2}, z_{n+1}) + d(z_n, z_{n+1}, a) + d(z_{n+1}, z_{n+2}, a)$
$\leq d(z_n, z_{n+2}, z_{n+1}) + \sum_{r=0}^{1} d(z_{n+r}, z_{n+r+1}, a)$

$$\therefore \int_0^{d(z_n,z_{n+2},a)} \emptyset(t)dt) \leq \int_0^{d(z_n,z_{n+2},z_{n+1})} \emptyset(t)dt + \sum_{r=0}^{1} \int_0^{d(z_{n+r},z_{n+r+1},a)} \emptyset(t)dt$$

Here we consider two possible cases to show that $\int_0^{d(z_n,z_{n+2},a)} \emptyset(t)dt = 0$.

Case I: $n = even = 2m$ (say)

$$\int_0^{d(z_n,z_{n+2},z_{n+1})} \emptyset(t)dt = \int_0^{d(z_{2m},z_{2m+2},z_{2m+1})} \emptyset(t)dt$$

$= \int_0^{d(z_{2m+2},z_{2m+1},z_{2m})} \emptyset(t)dt$

$= \int_0^{d(Jx_{2m+1}, Ix_{2m}, z_{2m})} \emptyset(t)dt$
$= \int_0^{d(Ix_{2m}, Jx_{2m+1}, z_{2m})} \emptyset(t)dt$

Now,

$$m_1(x_{2m}, x_{2m+1}, z_{2m}) = d(STx_{2m+1}, Jx_{2m+1}, z_{2m}) \left[\frac{1 + d(ABx_{2m}, Ix_{2m}, z_{2m})}{1 + d(ABx_{2m}, STx_{2m+1}, z_{2m})}\right]$$

$= d(z_{2m+1}, z_{2m+2}, z_{2m}) \left[\frac{1+d(z_{2m}, z_{2m+1}, z_{2m})}{1+d(z_{2m}, z_{2m+1}, z_{2m})}\right]$
$= d(z_{2m+1}, z_{2m+2}, z_{2m})$

$$m_2(x_{2m}, x_{2m+1}, z_{2m}) = d(ABx_{2m}, Ix_{2m}, z_{2m}) \left[\frac{1 + d(STx_{2m+1}, Jx_{2m+1}, z_{2m})}{1 + d(ABx_{2m}, STx_{2m+1}, z_{2m})}\right]$$

$= d(z_{2m}, z_{2m+1}, z_{2m}) \left[\frac{1+d(z_{2m+1}, z_{2m+2}, z_{2m})}{1+d(z_{2m}, z_{2m+1}, z_{2m})}\right]$
$= 0$

$$m_3(x_{2m}, x_{2m+1}, z_{2m}) = \frac{d(Ix_{2m}, STx_{2m+1}, z_{2m}).\, d(Jx_{2m+1}, ABx_{2m}, z_{2m})}{1 + d(ABx_{2m}, STx_{2m+1}, z_{2m})}$$

$= \frac{d(z_{2m+1}, z_{2m+1}, z_{2m}).d(z_{2m+2}, z_{2m}, z_{2m})}{1+d(z_{2m}, z_{2m+1}, z_{2m})}$
$= 0$

$m_4(x_{2m}, x_{2m+1}, z_{2m}) = max\{d(ABx_{2m}, STx_{2m+1}, z_{2m}), d(ABx_{2m}, Ix_{2m}, z_{2m}),$
$d(STx_{2m+1}, Jx_{2m+1}, z_{2m}), \frac{1}{2}[d(Ix_{2m}, STx_{2m+1}, z_{2m})$
$+d(Jx_{2m+1}, ABx_{2m}, z_{2m})]\}$
$= max\{d(z_{2m}, z_{2m+1}, z_{2m}), d(z_{2m}, z_{2m+1}, z_{2m}),$
$d(z_{2m+1}, z_{2m+2}, z_{2m}), \frac{1}{2}[d(z_{2m+1}, z_{2m+1}, z_{2m})$
$+d(z_{2m+2}, z_{2m}, z_{2m})]\}$
$= d(z_{2m+1}, z_{2m+2}, z_{2m})$

$$\therefore \int_0^{d(z_n, z_{n+2}, z_{n+1})} \emptyset(t)dt = \int_0^{d(z_{2m}, z_{2m+2}, z_{2m+1})} \emptyset(t)dt$$

$= \int_0^{d(Ix_{2m}, Jx_{2m+1}, z_{2m})} \emptyset(t)dt$
$\leq \psi(max\{\int_0^{m_i(x_{2m}, x_{2m+1}, z_{2m})} \emptyset(t)dt: 1 \leq i \leq 4\})$
$= \psi(max\{\int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m})} \emptyset(t)dt, 0, 0,$
$\int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m})} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m})} \emptyset(t)dt)$
$< \int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m})} \emptyset(t)dt$

which is a contradiction.

Hence, $\int_0^{d(z_{2m}, z_{2m+2}, z_{2m+1})} \emptyset(t)dt = 0$

$\therefore \quad \int_0^{d(z_n, z_{n+2}, a)} \emptyset(t)dt \leq \sum_{r=0}^{1} \int_0^{d(z_{n+r}, z_{n+r+1}, a)} \emptyset(t)dt$

Case II: $n = odd = 2m + 1$ (say)

Therefore,

$\int_0^{d(z_n, z_{n+2}, z_{n+1})} \emptyset(t)dt = \int_0^{d(z_{2m+1}, z_{2m+3}, z_{2m+2})} \emptyset(t)dt$
$= \int_0^{d(z_{2m+3}, z_{2m+2}, z_{2m+1})} \emptyset(t)dt$
$= \int_0^{d(Ix_{2m+2}, Jx_{2m+1}, z_{2m+1})} \emptyset(t)dt$

$$\therefore \int_0^{d(Ix_{2m+2}, Jx_{2m+1}, z_{2m+1})} \emptyset(t)dt \leq \psi(max\{\int_0^{m_i(x_{2m+2}, x_{2m+1}, z_{2m+1})} \emptyset(t)dt: 1 \leq i \leq 4\})$$

where

$$m_1(x_{2m+2}, x_{2m+1}, z_{2m+1}) = d(STx_{2m+1}, Jx_{2m+1}, z_{2m+1}) \left[\frac{1 + d(ABx_{2m+2}, Ix_{2m+2}, z_{2m+1})}{1 + d(ABx_{2m+2}, STx_{2m+1}, z_{2m+1})}\right]$$

$= d(z_{2m+1}, z_{2m+2}, z_{2m+1}) \left[\frac{1+d(z_{2m+2}, z_{2m+3}, z_{2m+1})}{1+d(z_{2m+2}, z_{2m+1}, z_{2m+1})}\right]$
$= 0$

$$m_2(x_{2m+2}, x_{2m+1}, z_{2m+1}) = d(ABx_{2m+2}, Ix_{2m+2}, z_{2m+1})\left[\frac{1+d(STx_{2m+1}, Jx_{2m+1}, z_{2m+1})}{1+d(ABx_{2m+2}, STx_{2m+1}, z_{2m+1})}\right]$$
$$= d(z_{2m+2}, z_{2m+3}, z_{2m+1})\left[\frac{1+d(z_{2m+1}, z_{2m+2}, z_{2m+1})}{1+d(z_{2m+2}, z_{2m+1}, z_{2m+1})}\right]$$
$$= d(z_{2m+1}, z_{2m+2}, z_{2m+3})$$
$$m_3(x_{2m+2}, x_{2m+1}, z_{2m+1}) = \frac{d(Ix_{2m+2}, STx_{2m+1}, z_{2m+1}).d(Jx_{2m+1}, ABx_{2m+2}, z_{2m+1}}{1+d(ABx_{2m+2}, STx_{2m+1}, z_{2m+1}}$$
$$= \frac{d(z_{2m+3}, z_{2m+1}, z_{2m+1}).d(z_{2m+2}, z_{2m+2}, z_{2m+1})}{1+d(z_{2m+2}, z_{2m+1}, z_{2m+1})}$$
$$= 0$$
$$m_4(x_{2m+2}, x_{2m+1}, z_{2m+1}) = max\{d(ABx_{2m+2}, STx_{2m+1}, z_{2m+1}),$$
$$d(ABx_{2m+2}, Ix_{2m+2}, z_{2m+1}),$$
$$d(STx_{2m+1}, Jx_{2m+1}, z_{2m+1}),$$
$$\frac{1}{2}[d(Ix_{2m+2}, STx_{2m+1}, z_{2m+1})$$
$$+d(Jx_{2m+1}, ABx_{2m+2}, z_{2m+1})]\}$$
$$= max\{d(z_{2m+2}, z_{2m+1}, z_{2m+1}), d(z_{2m+2}, z_{2m+3}, z_{2m+1}),$$
$$d(z_{2m+1}, z_{2m+2}, z_{2m+1}), \frac{1}{2}[d(z_{2m+3}, z_{2m+1}, z_{2m+1})$$
$$+d(z_{2m+2}, z_{2m+2}, z_{2m+1})]\}$$
$$= d(z_{2m+2}, z_{2m+3}, z_{2m+1})$$

$$\therefore \int_0^{d(z_n, z_{n+2}, z_{n+1})} \emptyset(t)dt = \int_0^{d(z_{2m+1}, z_{2m+3}, z_{2m+2})} \emptyset(t)dt$$
$$\leq \psi(max\{\int_0^{m_i(x_{2m+2}, x_{2m+1}, z_{2m+1})} \emptyset(t)dt: 1 \leq i \leq 4\})$$
$$= \psi(max\{0, \int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m+3})} \emptyset(t)dt, 0,$$
$$\int_0^{d(z_{2m+2}, z_{2m+3}, z_{2m+1})} \emptyset(t)dt\})$$
$$= \psi(\int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m+3})} \emptyset(t)dt)$$
$$< \int_0^{d(z_{2m+1}, z_{2m+2}, z_{2m+3})} \emptyset(t)dt$$

which is a contradiction.

Hence, $\int_0^{d(z_n, z_{n+2}, z_{n+1})} \emptyset(t)dt = 0$ when n is odd.

So, in either case $\int_0^{d(z_n, z_{n+2}, z_{n+1})} \emptyset(t)dt = 0$.

Therefore,

$$\int_0^{d(z_n, z_{n+2}, a)} \emptyset(t)dt) \leq \sum_{r=0}^{1} \int_0^{d(z_{n+r}, z_{n+r+1}, a)} \emptyset(t)dt$$

Proceeding in the same fashion, we get for any $p > 0$

$$\int_0^{d(z_n, z_{n+p}, a)} \emptyset(t)dt \leq \sum_{r=0}^{p-1} \int_0^{d(z_{n+r}, z_{n+r+1}, a)} \emptyset(t)dt \to 0$$

as $n \to \infty,\ p > 0$ [by (8)]

or, $\int_0^{d(z_n, z_{n+p}, a)} \emptyset(t)dt \to 0$ as $n \to \infty$.

Hence, $\lim_{n\to\infty} d(z_n, z_{n+p}, a) = 0$

or, $d(z_n, z_{n+p}, a) \to 0$ as $n \to \infty$.

Hence, $\{z_n\}_{n\in N}$ is a Cauchy sequence.

Let $AB(X)$ is complete. Since, $\{z_{2n}\}_{n\in N} \subseteq AB(X)$, which implies that $\{z_{2n}\}_{n\in N}$ converges to a point $z \in AB(X)$. Clearly, $\lim_{n\to\infty} z_n = z$. Put $u \in AB^{-1}z$. It follows that $ABu = z$.

If possible, let $Iu \neq z$ we have

$$m_1(u, x_{2n-1}, a) = d(STx_{2n-1}, Jx_{2n-1}, a)\left[\frac{1+d(ABu, Iu, a)}{1+d(ABu, STx_{2n-1}, a)}\right]$$
$$= d(z_{2n-1}, z_{2n}, a)\left[\frac{1+d(z, Iu, a)}{1+d(z, z_{2n-1}, a)}\right]$$
$$\to 0 \text{ as } n \to \infty$$
$$m_2(u, x_{2n-1}, a) = d(ABu, Iu, a)\left[\frac{1+d(STx_{2n-1}, Jx_{2n-1}, a)}{1+d(ABu, STx_{2n-1}, a)}\right]$$
$$= d(z, Iu, a)\left[\frac{1+d(z_{2n-1}, z_{2n}, a)}{1+d(z, z_{2n-1}, a)}\right]$$
$$\to d(z, Iu, a) \text{ as } n \to \infty$$

$m_3(u, x_{2n-1}, a) = \frac{d(Iu,STx_{2n-1},a).d(Jx_{2n-1},ABu,a)}{1+d(ABu,STx_{2n-1},a}$
$= \frac{d(Iu,z_{2n-1},a).d(z_{2n},z,a)}{1+d(z,z_{2n-1},a)}$
$\to 0$ as $n \to \infty$
$m_4(u, x_{2n-1}, a) = max\{d(ABu, STx_{2n-1}, a), d(ABu, Iu, a),$
$d(STx_{2n-1}, Jx_{2n-1}, a), \frac{1}{2}[d(Iu, STx_{2n-1}, a)$
$+d(Jx_{2n-1}, ABu, a)]\}$
$= max\{d(z, z_{2n-1}, a), d(z, Iu, a), d(z_{2n-1}, z_{2n}, a),$
$\frac{1}{2}[d(Iu, z_{2n-1}, a) + d(z_{2n}, z, a)]\}$
$\to d(z, Iu, a)$ as $n \to \infty$

Therefore,
$\int_0^{d(Iu,z,a)} \emptyset(t)dt = \lim_{n\to\infty} Sup \int_0^{d(Iu,z_{2n},a)} \emptyset(t)dt$
$= \lim_{n\to\infty} Sup \int_0^{d(Iu,Jx_{2n-1},a)} \emptyset(t)dt$
$\leq \lim_{n\to\infty} Sup\, \psi(max\{\int_0^{m_i(u,x_{2n-1},a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$\leq \psi(\lim_{n\to\infty} Sup\, max\{\int_0^{m_i(u,x_{2n-1},a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$\leq \psi(\max\{0, \int_0^{d(z,Iu,a)} \emptyset(t)dt, 0, \int_0^{d(z,Iu,a)} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z,Iu,a)} \emptyset(t)dt)$
$< \int_0^{d(z,Iu,a)} \emptyset(t)dt$

which is a contradiction.

Therefore, $\int_0^{d(z,Iu,a)} \emptyset(t)dt = 0$

or, $d(z, Iu, a) = 0$ or, $z = Iu$ (a being arbitrary).

Hence, $z = Iu = ABu$ which shows that the pair (AB, I) has a coincidence point.

Therefore, $I(X) \subseteq ST(X)$ therefore $z \in ST(X)$ and $\ni a$ point $v \in ST^{-1}z$ i.e. $STv = z$.

Now, if possible, let $Jv \neq z$ we have

$m_1(u, v, a) = d(STv, Jv, a)\left[\frac{1+d(ABu,Iu,a)}{1+d(ABu,STv,a)}\right]$
$= d(z, Jv, a)\left[\frac{1+d(z,z,a)}{1+d(z,z,a)}\right]$
$= d(z, Jv, a)$
$m_2(u, v, a) = d(ABu, Iu, a)\left[\frac{1+d(STv,Jv,a)}{1+d(ABu,STv,a)}\right]$
$= d(z, z, a)\left[\frac{1+d(z,Jv,a)}{1+d(z,z,a)}\right]$
$= 0$
$m_3(u, v, a) = \frac{d(Iu,STv,a).d(Jv,ABv,a)}{1+d(ABu,STv,a)}$
$= \frac{d(z,z,a).d(Jv,z,a)}{1+d(z,z,a)}$
$= 0$
$m_4(u, v, a) = max\{d(ABu, STv, a), d(ABu, Iu, a),$
$d(STv, Jv, a), \frac{1}{2}[d(Iu, STv, a) + d(Jv, ABu, a)]\}$
$= max\{d(z, z, a), d(z, z, a), d(z, Jv, a), \frac{1}{2}[0 + d(Jv, z, a)]\}$
$= max\{0,0, d(z, Jv, a), \frac{1}{2}d(z, Jv, a)\}$
$= d(z, Jv, a)$

and
$\int_0^{d(z,Jv,a)} \emptyset(t)dt = \int_0^{d(Iu,Jv,a)} \emptyset(t)dt$
$\leq \psi(max\{\int_0^{m_i(u,v,a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$= \psi(\max\{\int_0^{d(z,Jv,a)} \emptyset(t)dt, 0,0, \int_0^{d(z,Jv,a)} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z,Jv,a)} \emptyset(t)dt)$
$< \int_0^{d(z,Jv,a)} \emptyset(t)dt$

which is a contradiction.
Therefore, $\int_0^{d(z,Jv,a)} \emptyset(t)dt = 0$
or, $d(z,Jv,a) = 0$ [by def. of $'\emptyset'$]
or, $z = Jv = STv$
which shows that the pair (ST,J) has a coincidence point.
Consequently, we have

$$ABu = Iu = z$$

and $$STv = Jv = z$$

which establishes the results (a) and (b).
Since, (AB,I) and (ST,J) are weakly compatible, therefore

$$ABz = AB(Iu) = I(ABu) = Iz$$
$$STz = ST(Jv) = J(STv) = Jz$$

If possible, let $z \neq Iz$ then

$$m_1(z,v,a) = d(STv,Jv,a)\left[\frac{1+d(ABz,Iz,a)}{1+d(ABz,STv,a)}\right]$$
$$= d(z,z,a)\left[\frac{1+d(Iz,Iz,a)}{1+d(Iz,z,a)}\right]$$
$$= 0$$
$$m_2(z,v,a) = d(ABz,Iz,a)\left[\frac{1+d(STv,Jv,a)}{1+d(ABz,STv,a)}\right]$$
$$= d(Iz,Iz,a)\left[\frac{1+d(z,z,a)}{1+d(Iz,z,a)}\right]$$
$$= 0$$
$$m_3(z,v,a) = \frac{d(Iz,STv,a).d(Jv,ABz,a)}{1+d(ABz,STv,a)}$$
$$= \frac{d(Iz,z,a).d(z,Iz,a)}{1+d(Iz,z,a)}$$
$$= \frac{d^2(z,Iz,a)}{1+d(Iz,z,a)}$$
$$m_4(z,v,a) = max\{d(ABz,STv,a), d(ABz,Iz,a),$$
$$d(STv,Jv,a), \tfrac{1}{2}[d(Iz,STv,a) + d(Jv,ABz,a)]\}$$
$$= max\{d(Iz,z,a), d(Iz,Iz,a), d(z,z,a), \tfrac{1}{2}[d(Iz,z,a) + d(z,Iz,a)]\}$$
$$= max\{d(Iz,z,a), 0,0, \tfrac{1}{2}.2d(z,Iz,a)\}$$
$$= d(Iz,z,a)$$

and

$$\int_0^{d(Iz,z,a)} \emptyset(t)dt = \int_0^{d(Iz,Jv,a)} \emptyset(t)dt$$
$$\leq \psi(max\{\int_0^{m_i(z,v,a)} \emptyset(t)dt : 1 \leq i \leq 4\})$$
$$= \psi(\max\{0,0, \int_0^{\frac{d^2(z,Iz,a)}{1+d(z,Iz,a)}} \emptyset(t)dt, \int_0^{d(Iz,z,a)} \emptyset(t)dt\})$$
$$= \psi(\int_0^{d(Iz,z,a)} \emptyset(t)dt)$$
$$< \int_0^{d(Iz,z,a)} \emptyset(t)dt$$

which is impossible.
Hence, $\int_0^{d(Iz,z,a)} \emptyset(t)dt = 0$
or, $d(Iz,z,a) = 0$ [by def. of $'\emptyset'$]
or, $Iz = z$ (a being arbitrary)
Consequently, $z = Iz = ABz$
Again, if possible, let $z \neq Jz$ then

$$m_1(z,z,a) = d(STz,Jz,a)\left[\frac{1+d(ABz,Iz,a)}{1+d(ABz,STz,a)}\right]$$
$$= d(Jz,Jz,a)\left[\frac{1+d(z,z,a)}{1+d(z,Jz,a)}\right]$$
$$= 0$$
$$m_2(z,z,a) = d(ABz,Iz,a)\left[\frac{1+d(STz,Jz,a)}{1+d(ABz,STz,a)}\right]$$
$$= d(z,z,a)\left[\frac{1+d(Jz,Jz,a)}{1+d(z,Jz,a)}\right]$$
$$= 0$$

$m_3(z,z,a) = \frac{d(Iz,STz,a).d(Jz,ABz,a)}{1+d(ABz,STz,a)}$
$= \frac{d(z,Jz,a).d(Jz,z,a)}{1+d(z,Jz,a)}$
$= \frac{d^2(z,Jz,a)}{1+d(z,Jz,a)}$
$m_4(z,z,a) = max\{d(ABz,STz,a), d(ABz,Iz,a),$
$d(STz,Jz,a), \frac{1}{2}[d(Iz,STz,a) + d(Jz,ABz,a)]\}$
$= max\{d(z,Jz,a), d(z,Jz,a), d(Jz,Jz,a), \frac{1}{2}[d(z,Jz,a)$
$+d(Jz,z,a)]\}$
$= d(z,Jz,a)$

and
$\int_0^{d(z,Jz,a)} \emptyset(t)dt = \int_0^{d(Iz,Jz,a)} \emptyset(t)dt$
$\leq \psi(max\{\int_0^{m_i(z,z,a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$= \psi(\max\{0,0, \int_0^{\frac{d^2(z,Jz,a)}{1+d(z,Jz,a)}} \emptyset(t)dt, \int_0^{d(z,Jz,a)} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z,Jz,a)} \emptyset(t)dt)$
$< \int_0^{d(z,Jz,a)} \emptyset(t)dt$
which is a contradiction.
Hence, $\int_0^{d(z,Jz,a)} \emptyset(t)dt = 0$
or, $d(z,Jz,a) = 0$ [by def. of $'\emptyset'$]
or, $z = Jz$ (a being arbitrary)
Hence, $z = Jz = STz$
Consequently, $ABz = Iz = STz = Jz = z$ and z is a common fixed point of AB, ST, I and J.
To prove that z is unique, let z' be another common fixed point of AB, ST, I and J. Then
$m_1(z,z',a) = d(STz',Jz',a)\left[\frac{1+d(ABz,Iz,a)}{1+d(ABz,STz',a)}\right]$
$= d(z',z',a)\left[\frac{1+d(z,z,a)}{1+d(z,z',a)}\right]$
$= 0$
$m_2(z,z',a) = d(ABz,Iz,a)\left[\frac{1+d(STz',Jz',a)}{1+d(ABz,STz',a)}\right]$
$= d(z,z,a)\left[\frac{1+d(z',z',a)}{1+d(z,z',a)}\right]$
$= 0$
$m_3(z,z',a) = \left[\frac{d(Iz,STz',a).d(Jz',ABz,a)}{1+d(ABz,STz',a)}\right]$
$= \frac{d(z,z',a).d(z',z,a)}{1+d(z,z',a)}$
$= \frac{d^2(z,z',a)}{1+d(z,z',a)}$

$m_4(z,z',a) = max\{d(ABz,STz',a), d(ABz,Iz,a),$
$d(STz',Jz',a), \frac{1}{2}[d(Iz,STz',a) + d(Jz',ABz,a)]\}$
$= max\{d(z,z',a), d(z,z,a), d(z',z',a), \frac{1}{2}[d(z,z',a)$
$+d(z',z,a)]\}$
$= max\{d(z,z',a), 0,0, d(z,z',a)\}$
$= d(z,z',a)$
and
$\int_0^{d(z,z',a)} \emptyset(t)dt = \int_0^{d(Iz,Jz',a)} \emptyset(t)dt$
$\leq \psi(max\{\int_0^{m_i(z,z',a)} \emptyset(t)dt : 1 \leq i \leq 4\})$
$\leq \psi(\max\{0,0, \int_0^{\frac{d^2(z,z',a)}{1+d(z,z',a)}} \emptyset(t)dt, \int_0^{d(z,z',a)} \emptyset(t)dt\})$
$= \psi(\int_0^{d(z,z',a)} \emptyset(t)dt)$

$< \int_0^{d(z,z',a)} \emptyset(t)dt$

which is a contradiction.

Hence, $\int_0^{d(z,z',a)} \emptyset(t)dt = 0$

or, $d(z,z',a) = 0$ [by def. of $'\emptyset'$]

or, $z = z'$ (a being arbitrary)

Hence, AB, ST, I and J have at most a common fixed point z in X.

Finally, we prove that z is also a common fixed point of A, B, S, T, I and J. Let both the pairs (AB, I) and (ST, J) have a unique common fixed point in z.

Then,on using the commutativity of $(A, B), (A, I)$ and (B, I), we have

$$Az = A(ABz) = A(BAz) = AB(Az)$$
$$Az = A(Iz) = I(Az)$$
$$Bz = B(ABz) = B\big(A(Az)\big) = BA(Bz) = AB(Bz)$$
$$Bz = B(Iz) = I(Bz)$$

which implies that (AB, I) has common fixed points which are Az and Bz.

We get, thereby $Az = z = Bz = Iz = ABz$.

Similarly, using the commutativity of $(S,T), (S,J)$ and (I,J), $Sz = z = Tz = Jz = STz$ can be shown. Now, we need to show that $Az = Sz$ $(Bz = Iz)$.

We have

$$m_1(Az, Sz, a) = d(ST(Sz), J(Sz), a)\left[\frac{1+d(AB(Az),I(Az),a)}{1+d(AB(Az),ST(Sz),a)}\right]$$
$$= d(Sz, Sz, a)\left[\frac{1+d(Az,Az,a)}{1+d(Az,Sz,a)}\right]$$
$$= 0$$
$$m_2(Az, Sz, a) = d(AB(Az), I(Az), a)\left[\frac{1+d(ST(Sz),J(Sz),a)}{1+d(AB(Az),ST(Sz),a)}\right]$$
$$= d(Az, Az, a)\left[\frac{1+d(Sz,Sz,a)}{1+d(Az,Sz,a)}\right]$$
$$= 0$$
$$m_3(Az, Sz, a) = \frac{d(I(Az),ST(Sz),a).d(J(Sz),AB(Az),a)}{1+d(AB(Az),ST(Sz),a)}$$
$$= \frac{d(Az,Sz,a).d(Sz,Az,a)}{1+d(Az,Sz,a)}$$
$$= \frac{d^2(Az,Sz,a)}{1+d(Az,Sz,a)}$$
$$m_4(Az, Sz, a) = max\{d(AB(Az), ST(Sz), a), d(AB(Az), I(Az), a),$$
$$d(ST(Sz), J(Sz), a), \tfrac{1}{2}[d(I(Az), ST(Sz), a)$$
$$+d(J(Sz), AB(Az), a)]\}$$
$$= max\{d(Az, Sz, a), d(Az, Az, a), d(Sz, Sz, a),$$
$$\tfrac{1}{2}[d(Az, Sz, a) + d(Sz, Az, a)]\}$$
$$= max\{d(Az, Sz, a), 0,0, \tfrac{1}{2}.2d(Az, Sz, a)\}$$
$$= d(Az, Sz, a)$$

and

$$\int_0^{d(Az,Sz,a)} \emptyset(t)dt = \int_0^{d(I(Az),J(Sz),a)} \emptyset(t)dt$$
$$\le \psi(max\{\int_0^{m_i(Az,Sz,a)} \emptyset(t)dt : 1 \le i \le 4\})$$
$$= \psi(max\{0,0, \int_0^{\frac{d^2(Az,Sz,a)}{1+d(Az,Sz,a)}} \emptyset(t)dt \,, \int_0^{d(Az,Sz,a)} \emptyset(t)dt\})$$
$$= \psi(\int_0^{d(Az,Sz,a)} \emptyset(t)dt)$$
$$< \int_0^{d(Az,Sz,a)} \emptyset(t)dt$$

which is a contradiction.

Hence, $\int_0^{d(Az,Sz,a)} \emptyset(t)dt = 0$

or, $d(Az, Sz, a) = 0$ [by def. of $'\emptyset'$]

or, $Az = Sz$ (a being arbitrary)

Similarly, $Bz = Tz$ can be shown.

Thus, z is a unique common fixed point of A, B, S, T, I and J.

Our first corollary is obtained by putting AB=A, ST=B, I=S and J=T in theorem (2.1) which is a generalization of result of Liu et al. [130] from metric space to 2–metric space.

Corollary 2.2:

Let (X,d) be a complete 2–metric space. Let A, B, S and T be self mappings on (X,d) satisfying the following conditions

(i) $T(X) \subseteq A(X), S(X) \subseteq B(X)$

(ii) $\int_0^{d(Sx,Ty,a)} \emptyset(t)dt \leq \psi(max\{\int_0^{m_i(x,y,a)} \emptyset(t)dt : 1 \leq i \leq 4\}), \quad \forall x,y,a \in X$ where $\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of R^+ and $\int_0^{\epsilon} \emptyset(t)dt > 0$ for each $\epsilon > 0$. $\psi: R^+ \to R^+$ is non–negative and non–decreasing on R^+, $\psi(t) < t$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$ and

$$m_1(x,y,a) = d(By,Ty,a)\left[\frac{1+d(Ax,Sx,a)}{1+d(Ax,By,a)}\right]$$

$$m_2(x,y,a) = d(Ax,Sx,a)\left[\frac{1+d(By,Ty,a)}{1+d(Ax,By,a)}\right]$$

$$m_3(x,y,a) = \frac{d(Sx,By,a).d(Ty,Ax,a)}{1+d(Ax,By,a)}$$

$$m_4(x,y,a) = max\{d(Ax,By,a), d(Ax,Sx,a), d(By,Ty,a), \tfrac{1}{2}[d(Sx,By,a) + d(Ty,Ax,a)]\}$$

(iii) One of $A(X), B(X), S(X)$ and $T(X)$ is a complete subset of X.

(a) (A,S) has a coincidence point

(b) (B,T) has a coincidence point

Moreover, if the pairs (A,S) and (B,T) are coincidentally commuting (weakly compatible), then A, B, S and T have a unique common fixed point in X.

Our next corollary is obtained by putting $AB = ST =$Identity map, $I = S, J = T$ and $m_1(x,y,a) = m_2(x,y,a) = m_3(x,y,a) = 0$ in theorem (2.1) which generalizes the result of Vijayaraju et al. [18] from a metric space to a 2–metric space.

Corollary 2.3:

Let (X,d) be a complete 2–metric space and let S, T be self maps on X such that for each distinct $x,y \in X$

$$\int_0^{d(Sx,Ty,a)} \emptyset(t)dt \leq \psi(\int_0^{M(x,y,a)} \emptyset(t)dt)$$

where,

$$M(x,y,a) = max\left\{d(x,y,a), d(x,Sx,a), d(y,Ty,a), \left[\frac{d(x,Ty,a)+d(y,Sx,a)}{2}\right]\right\}$$

and $\emptyset: R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of R^+ and $\int_0^{\epsilon} \emptyset(t)dt > 0$ for each $\epsilon > 0$. $\psi: R^+ \to R^+$ is non–negativeand non–decreasing on R^+, $\psi(t) < t$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$then S and T have a unique common fixed point.

III. CONCLUSION:

Branciari [3] introduced the notion of contractions of integral type and proved fixed point theorem for this class of mappings. Further results on this class of mappings were obtained by Rhoades [15] and many others. Altun et al. [1] proved a fixed point theorem for weakly compatible mappings satisfying a general contractive condition of integral type. In this paperwe prove a common fixed point theorem of integral type contraction for weakly compatible mappings in complete 2–metric spaces. In this process, many known results are enriched and improved.

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